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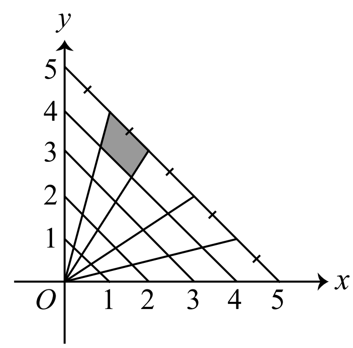
June 2017 Solutions

In honour of the 50th anniversary of the Faculty of Mathematics, at the beginning of each month of 2017, a set of five problems from the 54 years of CEMC contests will be posted. Solutions to the problems will be posted at the beginning of the next month. Hopefully, these problems will intrigue and inspire your mathematical mind. For more problem solving resources, please visit `cemc.uwaterloo.ca`.

1. *1990 Cayley Contest, Question 22*

   The five marked segments are equal in length. The area of the shaded region is

   (A) 0.5 **(B)** 0.9 **(C)** 1.0
   (D) 1.1 **(E)** 1.8



   Solution

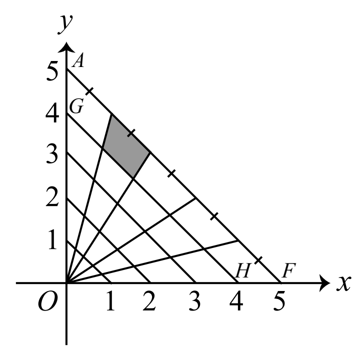
   We label points A, G, H, F.

   The area of quadrilateral $AGHF$ equals the difference between the areas of $\triangle OAF$ and $\triangle OGH$.

   Since AF is divided into five equal segments, then GH is also divided into five equal segments. (Can you see why?)

   Therefore, the area of the shaded quadrilateral (which is a trapezoid) is $\frac{1}{5}$ of the area of quadrilateral (trapezoid) $AGHF$. This is because the heights are equal and the two parallel bases of the shaded trapezoid are equal to $\frac{1}{5}$ of the corresponding parallel bases of $AGHF$.

   Therefore, the area of the shaded quadrilateral is $\frac{1}{5} \cdot \frac{9}{2} = \frac{9}{10}$.



ANSWER: (B)

2. *2002 Descartes Contest, Question A5*

   If $0^\circ < x < 90^\circ$ and $\tan(2x) = -\frac{24}{7}$, determine the value of $\sin x$.

   Solution

   Since $0^\circ < x < 90^\circ$, then $\tan x > 0$.

   Since $\tan(2x) = \dfrac{2\tan x}{1-\tan^2 x}$, then $\dfrac{2\tan x}{1-\tan^2 x} = -\dfrac{24}{7}$ gives $14\tan x = 24\tan^2 x - 24$.

   Rearranging and factoring, we obtain $12\tan^2 - 7\tan x - 12 = 0$ and $(4\tan x + 3)(3\tan x - 4) = 0$ and so $\tan x = -\dfrac{3}{4}$ or $\tan x = \dfrac{4}{3}$.

   Since $\tan x > 0$, then $\tan x = \dfrac{4}{3}$. From this we obtain $\dfrac{\sin x}{\cos x} = \dfrac{4}{3}$, and so $3\sin x = 4\cos x$.

   Squaring both sides, we obtain $9\sin^2 x = 16\cos^2 x = 16(1 - \sin^2 x)$.

   Rearranging, we obtain $25\sin^2 x = 16$ and so $\sin^2 x = \dfrac{16}{25}$ which gives $\sin x = \pm\dfrac{4}{5}$.

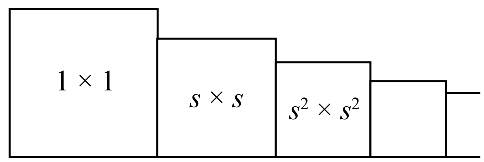
   Since $0^\circ < x < 90^\circ$, then $\sin x > 0$ which gives $\sin x = \dfrac{4}{5}$.

3. *2007 Cayley Contest, Question 20*
What is the largest integer n for which $3(n^{2007}) < 3^{4015}$?
(A) 2 **(B)** 3 **(C)** 6 **(D)** 8 **(E)** 9

Solution
Since $3(n^{2007}) < 3^{4015}$, then $n^{2007} < \frac{1}{3} \cdot 3^{4015} = 3^{4014}$.
But $3^{4014} = (3^2)^{2007} = 9^{2007}$ so we have $n^{2007} < 9^{2007}$.
Therefore, $n < 9$ and so the largest integer n that works is $n = 8$.

ANSWER: (D)

4. *1994 Descartes Contest, Question 7*
As shown below, a figure consists of an infinite sequence of squares which have sides of length 1, s, s^2, s^3, ..., where $0 < s < 1$.



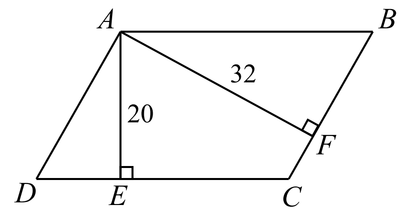
(a) Let A be the area of the figure. Express A in terms of s.
(b) Let P be the outside perimeter of the figure. Express P in terms of S.
(c) Determine all s such that $\dfrac{A}{P} = \dfrac{8}{35}$.

Solution

(a) The area of the figure is given by $A = 1 + s^2 + s^4 + s^6 + ... = \dfrac{1}{1-s^2}$.
This is an infinite geometric sum with first term $a = 1$ and common ratio $r = s^2$ and with $0 < s^2 < 1$.
(b) The length on the left side of the figure is 1.
The sum of the vertical segments on the right sides of the squares is 1.
The sum of the horizontal segments is $1 + s + s^2 + ...$ for both the top and the bottom.
Therefore, the perimeter equals
$$P = 2 + 2(1 + s + s^2 + s^3 + ...) = 2 + 2\left(\frac{1}{1-s}\right) = \frac{4-2s}{1-s}$$
(c) Using the information from (a) and (b), the following equations are equivalent:
$$\begin{aligned}
\frac{A}{P} &= \frac{8}{35} \\
35A &= 8P \\
\frac{35}{1-s^2} &= \frac{8(4-2s)}{1-s} \\
35(1-s) &= 8(4-2s)(1-s)(1+s) \\
35 &= 8(4-2s)(1+s) \qquad \text{(since } 1-s \neq 0\text{)} \\
35 &= -16s^2 + 16s + 32 \\
16s^2 - 16s + 3 &= 0 \\
(4s-3)(4s-1) &= 0
\end{aligned}$$
and so $s = \frac{1}{4}$ or $s = \frac{3}{4}$.

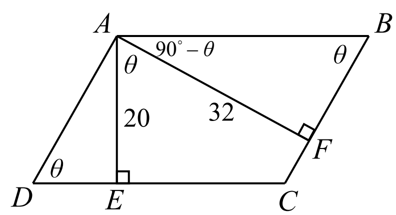
5. *2016 Euclid Contest, Question 8a*

In the diagram, $ABCD$ is a parallelogram. Point E is on DC with AE perpendicular to DC, and point F is on CB with AF perpendicular to CB. If $AE = 20$, $AF = 32$, and $\cos(\angle EAF) = \frac{1}{3}$, determine the exact value of the area of quadrilateral $AECF$.



Solution

Let $\angle EAF = \theta$.
Since $ABCD$ is a parallelogram, then AB and DC are parallel with $AB = DC$, and DA and CB are parallel with $DA = CB$.
Since AE is perpendicular to DC and AB and DC are parallel, then AE is perpendicular to AB.
In other words, $\angle EAB = 90^\circ$, and so $\angle FAB = 90^\circ - \theta$.
Since $\triangle AFB$ is right-angled at F and $\angle FAB = 90^\circ - \theta$, then $\angle ABF = \theta$.
Using similar arguments, we obtain that $\angle DAE = 90^\circ - \theta$ and $\angle ADE = \theta$.



Since $\cos(\angle EAF) = \cos\theta = \frac{1}{3}$ and $\cos^2\theta + \sin^2\theta = 1$, then

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{1}{9}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

(Note that $\sin\theta > 0$ since θ is an angle in a triangle.)
In $\triangle AFB, \sin\theta = \frac{AF}{AB}$ and $\cos\theta = \frac{FB}{AB}$.
Since $AF = 32$ and $\sin\theta = \frac{2\sqrt{2}}{3}$, then $AB = \frac{AF}{\sin\theta} = \frac{32}{2\sqrt{2}/3} = \frac{48}{\sqrt{2}} = 24\sqrt{2}$.
Since $AB = 24\sqrt{2}$ and $\cos\theta = \frac{1}{3}$, then $FB = AB\cos\theta = 24\sqrt{2}(\frac{1}{3}) = 8\sqrt{2}$.
In $\triangle AED, \sin\theta = \frac{AE}{AD}$ and $\cos\theta = \frac{DE}{AD}$.
Since $AE = 20$ and $\sin\theta = \frac{2\sqrt{2}}{3}$, then $AD = \frac{AE}{\sin\theta} = \frac{20}{2\sqrt{2}/3} = \frac{30}{\sqrt{2}} = 15\sqrt{2}$.
Since $AD = 15\sqrt{2}$ and $\cos\theta = \frac{1}{3}$, then $DE = AD\cos\theta = 15\sqrt{2}(\frac{1}{3}) = 5\sqrt{2}$.
(To calculate AD and DE, we could also have used the fact that $\triangle ADE$ is similar to $\triangle ABF$.)
Finally, the area of quadrilateral $AECF$ equals the area of parallelogram $ABCD$ minus the combined areas of $\triangle AFB$ and $\triangle ADE$.
The area of parallelogram $ABCD$ equals $AB \times AE = 24\sqrt{2} \times 20 = 480\sqrt{2}$.
The area of $\triangle AFB$ equals $\frac{1}{2}(AF)(FB) = \frac{1}{2}(32)(8\sqrt{2}) = 128\sqrt{2}$.
The area of $\triangle AED$ equals $\frac{1}{2}(AE)(DE) = \frac{1}{2}(20)(5\sqrt{2}) = 50\sqrt{2}$.
Thus, the area of quadrilateral $AECF$ is $480\sqrt{2} - 128\sqrt{2} - 50\sqrt{2} = 302\sqrt{2}$.