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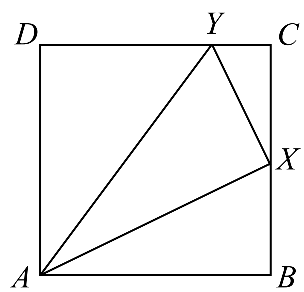
In honour of the 50th anniversary of the Faculty of Mathematics, at the beginning of each month of 2017, a set of five problems from the 54 years of CEMC contests will be posted. Solutions to the problems will be posted at the beginning of the next month. Hopefully, these problems will intrigue and inspire your mathematical mind. For more problem solving resources, please visit `cemc.uwaterloo.ca`.

1. *2001 Grade 11 Invitational Mathematics Challenge, Question 3*
   Points X and Y are on sides BC and CD of square $ABCD$, as shown. The lengths of XY, AX and AY are 3, 4 and 5, respectively. Determine the side length of square $ABCD$.



Solution
Since $\triangle AXY$ has side lengths $3, 4, 5$ and $3^2 + 4^2 = 5^2$, then $\angle AXY = 90^\circ$.
Let the side length of the square be s. Thus, $AB = AD = BC = CD = s$.
Let $BX = x$. Since $BC = s$, then $CX = BC - BX = s - x$.
Let $\angle BAX = \theta$.
Since $\triangle ABX$ is right-angled at B, then $\angle AXB = 90^\circ - \theta$.
Since $\angle AXY = 90^\circ$, then $\angle YXC = 180^\circ - \angle AXY - \angle AXB = \theta$.
Since $\triangle XCY$ is right-angled at C, then $\angle XYC = 90^\circ - \theta$.
Therefore, $\triangle ABX$ is similar to $\triangle XCY$ (angle-angle), and so $\dfrac{AB}{BX} = \dfrac{XC}{CY}$.
Re-arranging, we obtain $CY = \dfrac{BX \cdot XC}{AB} = \dfrac{x(s-x)}{s}$.
By the Pythagorean Theorem in $\triangle ABX$, $s^2 + x^2 = 4^2$.
By the Pythagorean Theorem in $\triangle XCY$,

$$\begin{aligned}
(s-x)^2 + \left(\frac{x(s-x)}{s}\right)^2 &= 9 \\
s^2(s-x)^2 + x^2(s-x)^2 &= 9s^2 \\
\left(s^2 + x^2\right)(s-x)^2 &= 9s^2 \\
16(s-x)^2 &= 9s^2 \\
(s-x)^2 &= \left(\tfrac{3}{4}s\right)^2 \\
s - x &= \tfrac{3}{4}s \qquad \text{(since } x < s \text{ and } \tfrac{3}{4}s > 0\text{)} \\
x &= \tfrac{1}{4}s
\end{aligned}$$

Substituting back into the equation $s^2 + x^2 = 16$, we obtain $s^2 + (\frac{1}{4}s)^2 = 16$ which yields $\frac{17}{16}s^2 = 16$ and so $s = \frac{16\sqrt{17}}{17}$. Thus, the side length of the square is $\frac{16\sqrt{17}}{17}$.

2. *2015 Canadian Intermediate Mathematics Contest, Question B2*

Alistair, Conrad, Emma, and Salma compete in a three-sport race. They each swim 2 km, then bike 40 km, and finally run 10 km. Also, they each switch instantly from swimming to biking and from biking to running.

(a) Emma has completed $\frac{1}{13}$ of the total distance of the race. How many kilometers has she travelled?

(b) Conrad began the race at 8:00 a.m. and completed the swimming portion in 30 minutes. Conrad biked 12 times as fast as he swam, and ran 3 times as fast as he swam. At what time did he finish the race?

(c) Alistair and Salma also began the race at 8:00 a.m. Alistair finished the swimming portion in 36 minutes, and then biked at 28 km/h. Salma finished the swimming portion in 30 minutes, and then biked at 24 km/h. Alistair passed Salma during the bike portion. At what time did Alistair pass Salma?

Solution

(a) The total length of the race is $2 + 40 + 10 = 52$ km.
When Emma has completed $\frac{1}{13}$ of the total distance, she has travelled $\frac{1}{13} \times 52 = 4$ km.

(b) Since Conrad completed the 2 km swim in 30 minutes (which is half an hour), then his speed was $2 \div \frac{1}{2} = 4$ km/h.
Since Conrad biked 12 times as fast as he swam, then he biked at $12 \times 4 = 48$ km/h.
Since Conrad biked 40 km, then the bike portion took him $\frac{40}{48} = \frac{5}{6}$ hours.
Since 1 hour equals 60 minutes, then the bike portion took him $\frac{5}{6} \times 60 = 50$ minutes.
Since Conrad ran 3 times as fast as he swam, then he ran at $3 \times 4 = 12$ km/h.
Since Conrad ran 10 km, then the running portion took him $\frac{10}{12} = \frac{5}{6}$ hours. This is again 50 minutes.
Therefore, the race took him $30 + 50 + 50 = 130$ minutes, or 2 hours and 10 minutes.
Since Conrad began the race at 8:00 a.m., then he completed the race at 10:10 a.m.

(c) Suppose that Alistair passed Salma after t minutes of the race.
Since Alistair swam for 36 minutes, then he had biked for $t - 36$ minutes (or $\dfrac{t-36}{60}$ hours) when they passed.
Since Salma swam for 30 minutes, then she had biked for $t - 30$ minutes (or $\dfrac{t-30}{60}$ hours) when they passed.
When Alistair and Salma passed, they had travelled the same total distance.
At this time, Alistair had swum 2 km. Since he bikes at 28 km/h, he had biked $28 \times \dfrac{t-36}{60}$ km.
Similarly, Salma had swum 2 km. Since she bikes at 24 km/h, she had biked $24 \times \dfrac{t-30}{60}$ km.
Since their total distances are the same, then

$$\begin{aligned}
2 + 28 \times \frac{t-36}{60} &= 2 + 24 \times \frac{t-30}{60} \\
28 \times \frac{t-36}{60} &= 24 \times \frac{t-30}{60} \\
28(t-36) &= 24(t-30) \\
7(t-36) &= 6(t-30) \\
7t - 252 &= 6t - 180 \\
t &= 72
\end{aligned}$$

Therefore, Alistair passed Salma 72 minutes into the race.
Since the race began at 8:00 a.m., then he passed her at 9:12 a.m.

3. *1991 Fermat Contest, Question 21*
A hardware store sells single digits to be used for house numbers. There are five 5s, four 4s, three 3s, and two 2s available. From this selection of digits, a customer is able to purchase his three-digit house number. The number of possible house numbers this customer could have is

(A) 63 **(B)** 24 **(C)** 60 **(D)** 48 **(E)** 39

Solution
Suppose that there were indeed three or more 2s available.
For the first digit, the customer can choose from the digits 5, 4, 3, and 2.
Therefore, there are 4 choices for the first digit.
Similarly, there are 4 choices for the second digit and 4 choices for the third digit.
This gives $4 \times 4 \times 4 = 64$ possible three-digit house numbers that can be made.
But there are actually only two 2s available, so not all of these house numbers can be made. In particular, the house number 222 cannot be made, but all of the others made above can still be made.
Therefore, the customer could form $64 - 1 = 63$ different three-digit house numbers.

ANSWER: (A)

4. *1996 Pascal Contest, Question 25*
There are exactly k perfect squares which are divisors of 1996^{1996}. The sum of the digits in the number k is

(A) 29 **(B)** 26 **(C)** 30 **(D)** 22 **(E)** 27

Solution
Let $n = 1996^{1996}$.
Factoring 1996, we get $1996 = 2 \times 2 \times 499$. We note that 2 and 499 are both prime numbers.
Therefore, $n = 1996^{1996} = (2^2 \times 499)^{1996} = 2^{2\times 1996} \times 499^{1996}$.
So all of the divisors of n are of the form $2^p \times 499^q$ for some integers p and q with $0 \leq p \leq 2 \times 1996$ and $0 \leq q \leq 1996$.
For such a divisor to be a perfect square, it must have an *even* number of factors of both 2 and 499.
There are $\frac{2\times 1996}{2} + 1 = 1997$ even numbers less than or equal to 2×1996 (including 0), and there are $\frac{1996}{2} + 1 = 998 + 1 = 999$ even numbers less than or equal to 1996 (including 0).
Thus, there are 1997 choices for p and 999 choices for q, giving $k = 1997 \times 999 = 1995003$ divisors that are perfect squares.
The sum of the digits of k is $1 + 9 + 9 + 5 + 0 + 0 + 3 = 27$.

ANSWER: (E)

5. *2017 Euclid Contest Question, Question 4b*

In an arithmetic sequence with 5 terms, the sum of the squares of the first 3 terms equals the sum of the squares of the last 2 terms. If the first term is 5, determine all possible values of the fifth term.

(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, $3, 5, 7, 9, 11$ is an arithmetic sequence with five terms.)

Solution

Let the common difference in this arithmetic sequence be d.
Since the first term in the sequence is 5, then the 5 terms are $5, 5+d, 5+2d, 5+3d, 5+4d$.
From the given information, $5^2 + (5+d)^2 + (5+2d)^2 = (5+3d)^2 + (5+4d)^2$.
Manipulating, we obtain the following equivalent equations:

$$\begin{aligned} 5^2 + (5+d)^2 + (5+2d)^2 &= (5+3d)^2 + (5+4d)^2 \\ 25 + (25+10d+d^2) + (25+20d+4d^2) &= (25+30d+9d^2) + (25+40d+16d^2) \\ 75 + 30d + 5d^2 &= 50 + 70d + 25d^2 \\ 0 &= 20d^2 + 40d - 25 \\ 0 &= 4d^2 + 8d - 5 \\ 0 &= (2d+5)(2d-1) \end{aligned}$$

Therefore, $d = -\frac{5}{2}$ or $d = \frac{1}{2}$.
These give possible fifth terms of $5 + 4d = 5 + 4(-\frac{5}{2}) = -5$ and $5 + 4d = 5 + 4(\frac{1}{2}) = 7$.
(We note that, for these two values of d, the sequences are $5, \frac{5}{2}, 0, -\frac{5}{2}, -5$ and $5, \frac{11}{2}, 6, \frac{13}{2}, 7$.)