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November 2017 Solutions

In honour of the 50th anniversary of the Faculty of Mathematics, at the beginning of each month of 2017, a set of five problems from the 54 years of CEMC contests will be posted. Solutions to the problems will be posted at the beginning of the next month. Hopefully, these problems will intrigue and inspire your mathematical mind. For more problem solving resources, please visit `cemc.uwaterloo.ca`.

1. *1992 Euclid Contest, Question 2c*
   The parabola $y = x^2 - 6x + 8$ intersects the x-axis at points A and B and the vertex of the parabola is at point C. Determine the area of triangle ABC.

   Solution
   To determine the x-intercepts, we set $y = 0$ and obtain $x^2 - 6x + 8 = 0$.
   Factoring, we get $(x - 2)(x - 4) = 0$ and so $x = 2$ or $x = 4$.
   Therefore, we can let the coordinates of A and B be $A(2, 0)$ and $B(4, 0)$.
   Since the roots are $x = 2$ and $x = 4$, then the axis of symmetry of the parabola has equation $x = \frac{1}{2}(2 + 4) = 3$.
   Since the vertex lies on the axis of symmetry, then its x-coordinate is 3 and its y-coordinate is thus $y = 3^2 - 6(3) + 8 = -1$.
   We can view $\triangle ABC$ as having base AB (of length $4-2 = 2$) and height running from the x-axis down to $C(3, -1)$. (This means that the height has length 1.)
   Hence, the area of $\triangle ABC$ is $\frac{1}{2}(2)(1) = 1$.

2. *2009 Euclid Contest, Question 9a*
   If $\log_2 x$, $1 + \log_4 x$, and $\log_8 4x$ are consecutive terms of a geometric sequence, determine the possible values of x.

   (A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a constant. For example, $3, 6, 12$ is a geometric sequence with three terms.)

   Solution
   First, we convert each of the logarithms to a logarithm with base 2:

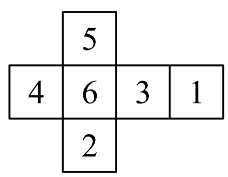
$$\begin{aligned} 1 + \log_4 x &= 1 + \frac{\log_2 x}{\log_2 4} = 1 + \frac{\log_2 x}{2} = 1 + \frac{1}{2}\log_2 x \\ \log_8 4x &= \frac{\log_2 4x}{\log_2 8} = \frac{\log_2 4 + \log_2 x}{3} = \frac{2}{3} + \frac{1}{3}\log_2 x \end{aligned}$$

   Let $y = \log_2 x$. Then the three terms are y, $1 + \frac{1}{2}y$, and $\frac{2}{3} + \frac{1}{3}y$. Since these three are in geometric sequence, then

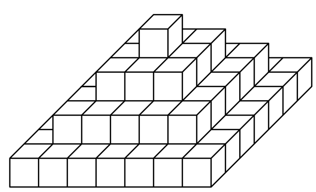
$$\begin{aligned}
\frac{y}{1+\frac{1}{2}y} &= \frac{1+\frac{1}{2}y}{\frac{2}{3}+\frac{1}{3}y} \\
y(\tfrac{2}{3}+\tfrac{1}{3}y) &= (1+\tfrac{1}{2}y)^2 \\
\tfrac{2}{3}y+\tfrac{1}{3}y^2 &= 1+y+\tfrac{1}{4}y^2 \\
8y+4y^2 &= 12+12y+3y^2 \\
y^2-4y-12 &= 0 \\
(y-6)(y+2) &= 0
\end{aligned}$$

Therefore, $y = \log_2 x = 6$ or $y = \log_2 x = -2$, which gives $x = 2^6 = 64$ or $x = 2^{-2} = \frac{1}{4}$.
When $x = 64$, the terms in the original sequence are $\log_2 64 = 6$, $1 + \log_4 64 = 4$, and $\log_8 256 = \frac{8}{3}$.
When $x = \frac{1}{4}$, the terms in the original sequence are $\log_2 \frac{1}{4} = -2$, $1 + \log_4 \frac{1}{4} = 0$, and $\log_8 1 = 0$.
Some definitions of a geometric sequence forbid the "common ratio" to be 0 and so exclude this second sequence as geometric.

3. *2006 Fryer Contest, Question 2*
Dmitri has a collection of identical cubes. Each cube is labelled with the integers 1 to 6 as shown in the following net:



(This net can be folded to make a cube.)
He forms a pyramid by stacking layers of the cubes on a table, as shown, with the bottom layer being a 7 by 7 square of cubes.



(a) Determine the total number of cubes used to build the pyramid. Explain how you got your answer.
(b) How many faces are visible after the pyramid is built and sitting on the table?
(c) Explain in detail how he should position the cubes so that if all of the visible numbers are added up, the total is as large as possible. What is this total?

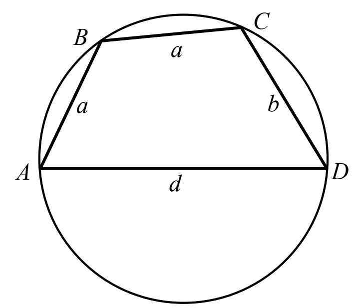
Solution

(a) The bottom layer of the cube is a 7 by 7 square of cubes, so uses $7 \times 7 = 49$ cubes.
The next layer of the cube is a 5 by 5 square of cubes, so uses $5 \times 5 = 25$ cubes.
The next layer of the cube is a 3 by 3 square of cubes, so uses $3 \times 3 = 9$ cubes.
The top layer consists of a single cube.
Therefore, the total number of cubes used is $49 + 25 + 9 + 1 = 84$.

(b) When we look at the pyramid from the top, we see a 7×7 square of visible faces, or 49 visible faces. (This square is composed of faces from all of the levels of the pyramid.)
When we look at the pyramid from each of the four sides, we see $1+3+5+7=16$ visible faces, so there are $4(16)=64$ visible faces on the sides.
Therefore, there are $49+64=113$ visible faces in total.

(c) To make the total of all of the visible numbers as large as possible, we should position the cubes so that the largest possible two, three or five numbers are visible, depending on its position.
For the top cube (with 5 visible faces), we position this cube with the "1" on the bottom face (and so is hidden). The total of the numbers visible on this cube is $2+3+4+5+6=20$.
For the 4 corner cubes on each layer (each with 3 visible faces), we position these cubes with the 4, 5 and 6 all visible (this is possible since the faces with the 4, 5 and 6 share a vertex) and the 1, 2 and 3 hidden. There are 12 of these cubes, so the total of the numbers visible on these cubes is $12 \times (4+5+6)$ or 180.
For the cubes on the sides (that is, not at the corner) of each layer, we position the cubes with the 5 and 6 visible (this is possible since the faces with 5 and 6 share an edge) and the 1, 2, 3, and 4 hidden. There are $4+12+20=36$ of these cubes, so the total of the numbers visible on these cubes is $36 \times (5+6)=396$.
Therefore, the overall largest possible total is $20+180+396=596$.

4. *1999 Euclid Contest, Question 10*
$ABCD$ is a cyclic quadrilateral, as shown, with side $AD=d$, where d is the diameter of the circle. $AB=a, BC=a$ and $CD=b$. If a, b, and d are integers $a \neq b$,

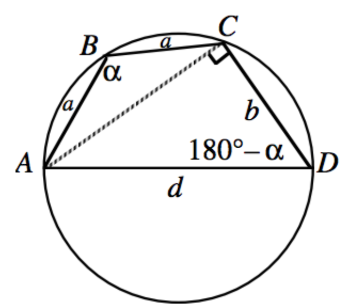


(a) prove that d cannot be a prime number.

(b) determine the *minimum* value of d.

Solution

(a) Join A to C. Since $\angle ACD$ is in a semicircle, $\angle ACD=90^\circ$.
Let $\angle ABC=\alpha$.
Since $ABCD$ is a cyclic quadrilateral, then $\angle CDA=180^\circ-\alpha$.
Using the cosine law in $\triangle ABC$, we obtain $AC^2=a^2+a^2-2a^2\cos\alpha$.
By the Pythagorean Theorem in $\triangle ACD$, $AC^2=d^2-b^2$.
Using trigonometric ratios in $\triangle ACD$, $\cos(180^\circ-\alpha)=\dfrac{b}{d}$.
Since $\cos(180^\circ-\alpha)=-\cos\alpha$, then $\cos\alpha=\dfrac{-b}{d}$.
By substitution,



$$
\begin{aligned}
d^2-b^2 &= 2a^2-2a^2\left(\frac{-b}{d}\right)\\
d^3-db^2 &= 2a^2d+2a^2b\\
d(d^2-b^2) &= 2a^2(d+b)\\
d(d+b)(d-b) &= 2a^2(d+b)\\
2a^2 &= d(d-b) \qquad \text{(since } d+b\neq 0\text{)}
\end{aligned}
$$

From here, we assume that d is prime and derive a contradiction.

Case 1: $d = 2$

If we make the substitution $d = 2$ intp $2a^2 = d(d - b)$ then we have the following equivalent equations:

$$\begin{aligned} 2a^2 &= 2(2 - b) \\ a^2 &= 2 - b \\ b + a^2 &= 2 \end{aligned}$$

Since a and b are integers then this implies $a = b = 1$ which is not possible since we are told that a and b must be different.

Case 2: d prime with $d \geq 3$

Start with $2a^2 = d(d - b)$.
Since d is a divisor of the right side, then d is a divisor of the left side.
Since $d \geq 3$, then d is not a divisor of 2, so d is a divisor of a^2.
Since d is prime and d is a divisor of a^2, then d is a divisor of a.
This means that $a \geq d$.
This is not possible, since d is the diameter of the circle and so $d > a$. (The diameter is the longest chord in a circle.)
Thus, the assumption that d is prime must be incorrect and so d must be a composite number.

(b) Note that d is not prime so $d \neq 2, 3, 5, 7$, etc.
Since a and b are integers and less than d, then $d \neq 1$.
To find the minimum possible value of d, we work our way through the smallest composite integers until we find one that works.

Suppose $d = 4$.
Then $2a^2 = 4(4 - b)$. This gives $a^2 = 2(4 - b)$.
If $b = 1$ or $b = 3$,then $a^2 = 6$ or $a^2 = 2$ so a is not an integer.
If $b = 2$ then $a = 2$ but $a \neq b$ so this is not possible.

Suppose $d = 6$.
Then $2a^2 = 6(6 - b)$. This gives $a^2 = 3(6 - b)$.
If $b = 1, 2, 4, 5$, a is not an integer.
If $b = 3$, then $a = 3$ but $a \neq b$ as before.

Suppose $d = 8$.
Then $2a^2 = 8(8 - b)$. This gives $a^2 = 4(8 - b)$.
If $b = 7$, then $a = 2$, which is an acceptable solution.
So the minimum possible value of d is 8.

5. *2017 Grade 8 Gauss Contest, Question 14*
There are 20 pens to be given away to 4 students. Each student receives a different number of pens and each student receives at least one pen. What is the largest number of pens that a student can receive?

(A) 17 **(B)** 15 **(C)** 14 **(D)** 8 **(E)** 5

Solution
If three of the students receive the smallest total number of pens possible, then the remaining student will receive the largest number of pens possible.
The smallest number of pens that a student can receive is 1, since each student receives at least 1 pen.
Since each student receives a different number of pens, the second smallest number of pens that a

student can receive is 2 and the third smallest number of pens that a student can receive is 3. The smallest total number of pens that three students can receive is $1 + 2 + 3 = 6$.
Therefore, the largest number of pens that a student can receive is $20 - 6 = 14$.