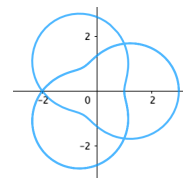# CEMC Math Circles - Grade 11/12

October 14 - 20, 2020
Polar Coordinates

The *Cartesian Coordinate System* is the most familiar system that we use to represent points in the plane. Today, we will learn about a different system, the *Polar Coordinate System*. In a few weeks, we will learn how to graph interesting curves like the two above. Equations for graphs like these are often very complicated using Cartesian coordinates, but can be much simpler using polar coordinates.

We will be using radians in this activity. If you have never measured angles in radians before, then check out this lesson *from the CEMC courseware.*

Warm-up: Radians

When we first learn about angles, we write their measures (that is, their "sizes") using degrees. For example, a complete circular angle measures $360°$, a straight angle measures $180°$, and a right angle measures $90°$. Angles like $30°$, $45°$, and $60°$ are also familiar.

A second way of measuring angles is in *radians*. In this case, a complete circular angle measures 2π. What connection can you see between 2π and the unit circle? The circumference of the unit circle is 2π. Radians are defined so that an angle of measure $x°$ measures $\dfrac{\pi x}{180}$ radians.

The value $\dfrac{\pi x}{180}$ *is actually the arc length of a sector of the unit circle defined by the angle with measure* $x°$ *so radians are in some sense measuring the arc length corresponding to the angle, which is one way of measuring the angle itself.*
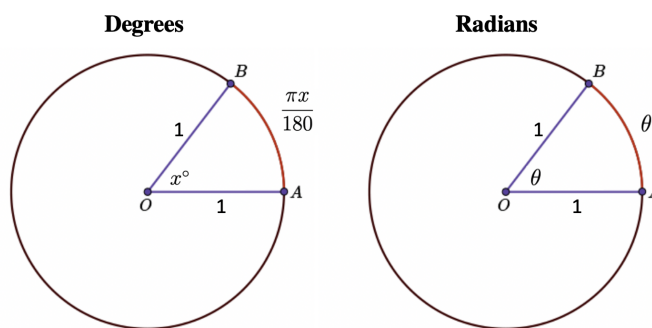
Questions:

(a) Convert the angles with the following measures from degrees to radians: $180°$, $90°$, $60°$, $45°$, $30°$, $48°$.

(b) Convert the angles with the following measures from radians to degrees: $\dfrac{\pi}{5}$, $\dfrac{5\pi}{6}$, $\dfrac{3\pi}{2}$, $\dfrac{7\pi}{4}$.

(c) Complete the chart below. *The angles are given in radians.*

| θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$					
$\cos\theta$					

Polar Coordinates

In Cartesian coordinates, a point P in the plane is given as $P(x, y)$, where x and y are real numbers.

Remind yourself of exactly what the values x and y represent here.

The point P can also be described using *polar coordinates* (r, θ). Here, r is the distance between the point P and the origin O. Also, θ is the angle (in radians) measured from the x-axis. (*Like when we look at the unit circle, positive angles are measured counter-clockwise from the positive x-axis.*) In polar coordinates, we call the positive x-axis the *polar axis*.
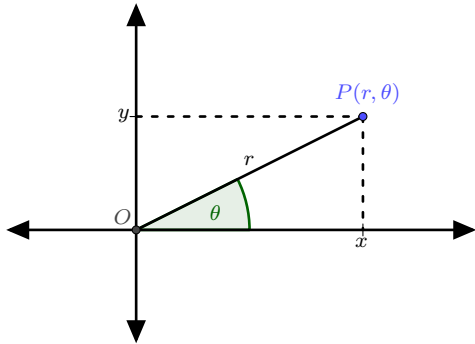
Suppose that P is in the first quadrant. Consider the right-angled triangle formed by the point P, the origin O, and the vertical line from P to the x-axis. This triangle has base x, height y and hypotenuse r.

By the Pythagorean Theorem, $r^2 = x^2 + y^2$.

We have $\cos\theta = \dfrac{x}{r}$ and $\sin\theta = \dfrac{y}{r}$ from the definitions of sine and cosine in right-angled triangles.

Manipulating these equations, we obtain the three equations below that help us to relate the polar and Cartesian coordinates of the point P.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2 + y^2}$$

Example

Consider the Cartesian point $Q(1, 1)$. Since $x = 1$ and $y = 1$, then $r = \sqrt{1^2 + 1^2} = \sqrt{2}$. Also, the line segment joining the origin O to Q makes an angle of $\frac{\pi}{4}$ with the positive x-axis. This means that polar coordinates for Q are $(\sqrt{2}, \frac{\pi}{4})$.

Try drawing a picture and clearly labelling x, y, r, and θ. Make sure you understand why these values of r and θ are correct.

Question 1

Plot the points with Cartesian coordinates $A(8\sqrt{3}, 8)$ and $B(\frac{5}{4}, \frac{5\sqrt{3}}{4})$ and then convert them to polar coordinates.

Question 2

Plot the points with Cartesian coordinates $C(8, -8\sqrt{3})$ and $D(-\frac{5\sqrt{3}}{4}, -\frac{5}{4})$ and then convert them to polar coordinates.

Example

Consider the point with polar coordinates $\left(4, \frac{3\pi}{2}\right)$. Since $r = 4$ and $\theta = \frac{3\pi}{2}$, we have that

$$x = r\cos\theta = 4\cos\left(\frac{3\pi}{2}\right) = 4(0) = 0$$

$$y = r\sin\theta = 4\sin\left(\frac{3\pi}{2}\right) = 4(-1) = -4$$

This means that the Cartesian coordinates of the point are $(0, -4)$.

Can you see why these must be the correct Cartesian coordinates by visualizing the point?

Note: *Before completing the following activity, check this* [*introductory video*](#) *to complement the previous reading.*

Activity

Consider the polar coordinates (r, θ), with $0 \le \theta < 2\pi$, of each of the 12 points plotted in the graph below. Exactly one of these points satisfies each of the following properties, and each point is labelled with a different letter. Determine which point best matches each property and use this information to complete the phrase below.

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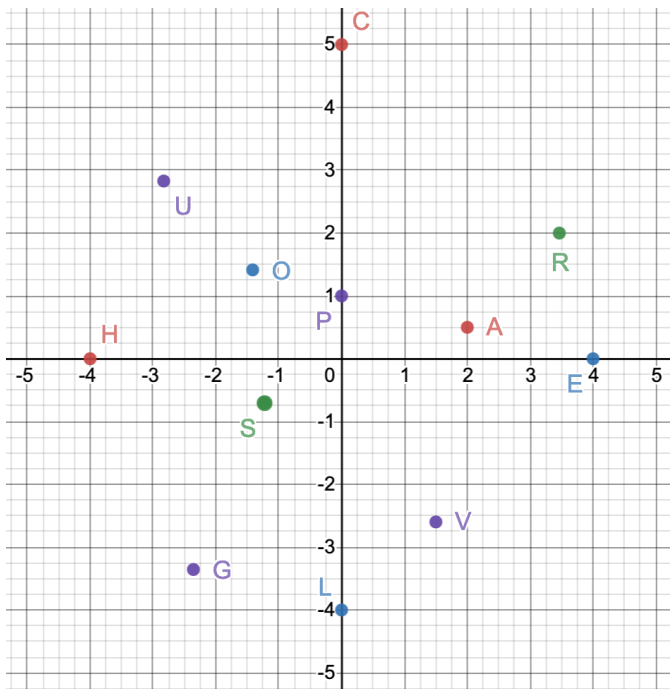
1. This point has polar coordinates $(4, 0)$.

2. This point has polar coordinates $(4, \frac{3\pi}{2})$.

3. This point has polar coordinates $(4, \frac{3\pi}{4})$.

4. This point could also be described using polar coordinates $(2, \frac{11\pi}{4})$.

5. This point's first coordinate, r, satisfies $r^2 = 2$.

6. This point has the largest first coordinate, r, out of all of the points.

7. This point has the smallest *positive* second coordinate, θ, out of all of the points.

8. This point's second coordinate, θ, satisfies $2\sin\theta = 1$.

9. This point's second coordinate, θ, satisfies $\cos\theta = -1$.

10. This point's first coordinate, r, satisfies $r = 3$.

11. This point's coordinates satisfy $r = \sin\theta$. *Remember that* $-1 \le \sin\theta \le 1$.

12. This point's coordinates satisfy $r = \theta$.

In a few weeks we will learn how to...

$\overline{\underset{12}{\hspace{1em}}}\ \overline{\underset{8}{\hspace{1em}}}\ \overline{\underset{7}{\hspace{1em}}}\ \overline{\underset{11}{\hspace{1em}}}\ \overline{\underset{9}{\hspace{1em}}}\ \quad \overline{\underset{11}{\hspace{1em}}}\ \overline{\underset{4}{\hspace{1em}}}\ \overline{\underset{2}{\hspace{1em}}}\ \overline{\underset{7}{\hspace{1em}}}\ \overline{\underset{8}{\hspace{1em}}}\ \quad \overline{\underset{6}{\hspace{1em}}}\ \overline{\underset{3}{\hspace{1em}}}\ \overline{\underset{8}{\hspace{1em}}}\ \overline{\underset{10}{\hspace{1em}}}\ \overline{\underset{1}{\hspace{1em}}}\ \overline{\underset{5}{\hspace{1em}}}$!

Can we plot points with polar coordinates with negative values of r?

We can extend the definition of polar coordinates to include negative values of r.

How do we interpret the polar coordinates $(1, \frac{\pi}{2})$ versus the polar coordinates $(-1, \frac{\pi}{2})$?

- The fact that they both have the same angle $\frac{\pi}{2}$ tells us that they both describe points that lie on the line passing through the origin and making an angle of $\frac{\pi}{2}$ with the positive x-axis.

- The magnitude of the radii both being 1 tell us that they both describe points that are 1 unit from the origin.

- The different signs tell us that they describe points on opposite sides of the origin. The negative means that we move in the direction *opposite* to the direction defined the ray $\theta = \frac{\pi}{2}$. This means moving in the direction defined by the ray $\theta = \frac{3\pi}{2}$.