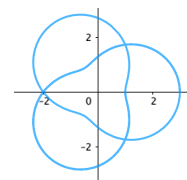


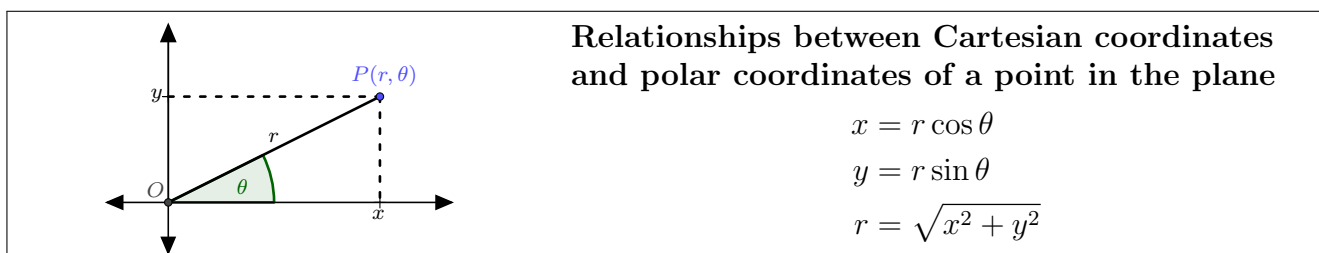
CEMC Math Circles - Grade 11/12

October 28 - November 3, 2020

Polar Curves



Two weeks ago we learned about a different coordinate system for the plane: the *Polar Coordinate System*. Remind yourself about how to work with polar coordinates before you try this activity.



Why might we want to view the plane through the lens of polar coordinates? One reason is that simple equations of the form $r = f(\theta)$ involving polar coordinates can lead to interesting graphs!

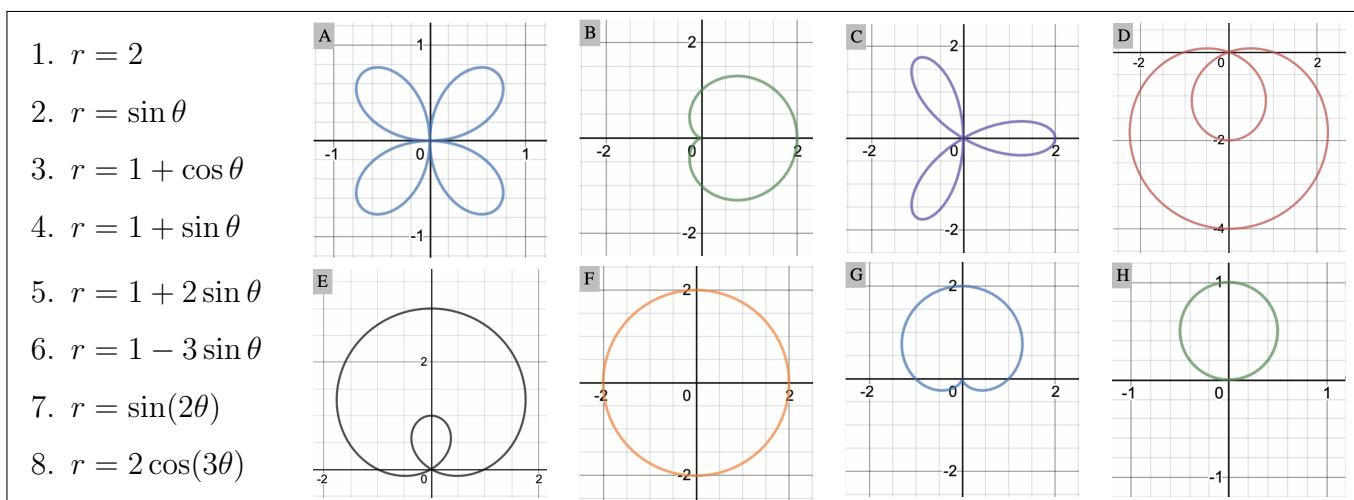
Let f be a function on the real numbers. The graph of the polar equation $r = f(\theta)$ consists of all points in the plane that have polar coordinates, (r, θ) , that satisfy the relation $r = f(\theta)$.

Note: Before working on the activity, read the examples and watch the explanations in [this video](#).

Activity

Consider the following polar equations and the graphs below. Exactly one of the graphs corresponds to each equation. Can you match each equation with its graph? Think about the following techniques:

- Plot some key points on the curve. For example, when $\theta = \frac{\pi}{2}$, what is the value of r ?
- Remember that $-1 \leq \sin \theta \leq 1$ and $-1 \leq \cos \theta \leq 1$. What does this mean for the range of r ?
- Think about how r changes as θ changes. (See the next pages for help with this.)
- How are points with a negative r -coordinate plotted? (We saw how to do this before.)



Example 1: Look at graph F. You should recognize this as a circle centred at the origin with radius 2. The points on this curve must be the points having polar coordinates that look like $(2, \theta)$ for some θ (2 units from the origin, at any angle). This means graph F must be matched with equation $r = 2$.

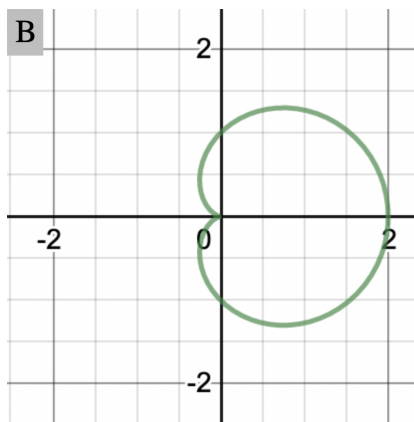


Note that we could also determine what the graph of $r = 2$ must look like by transforming this polar equation into a Cartesian equation. Since $r = \sqrt{x^2 + y^2}$, a point's polar coordinates satisfy the equation $r = 2$ exactly when its Cartesian coordinates satisfy the equation $\sqrt{x^2 + y^2} = 2$. Squaring both sides reveals the equation $x^2 + y^2 = 4$ which describes the circle shown!

Can you match each of the eight graphs with one of the eight equations without actually trying to sketch the complete graphs of the polar equations? Read the following example to get you started on possible matching strategies that do not involve graphing the polar equations.

Example 2

Consider graph B. Given that this graph is **matched with one of the five equations below**, can you figure out which one by eliminating all but one equation?



1. $r = 2$
2. $r = \sin \theta$
3. $r = 1 + \cos \theta$
4. $r = 1 + \sin \theta$
7. $r = \sin(2\theta)$

Let's see if we can use only the range of r to eliminate several possibilities.

1. Graph B cannot be the graph of $r = 2$: We have already determined that $r = 2$ is matched with another graph.
2. Graph B cannot be the graph of $r = \sin \theta$: Since $\sin \theta$ cannot be larger than 1, no points on the graph of this polar equation can be more than 1 unit from the origin. Graph B has at least one point 2 units from the origin.
3. Graph B might be the graph of $r = 1 + \cos \theta$: Since $-1 \leq \cos \theta \leq 1$, we have $0 \leq 1 + \cos \theta \leq 2$ and so the points on this graph should all be within 2 units of the origin or exactly 2 units from the origin. This is true of the graph B.
4. Graph B might be the graph of $r = 1 + \sin \theta$: Similar reasoning as in 3.
7. Graph B cannot be the graph of $r = \sin(2\theta)$: Similar reasoning as in 2.

By considering the range of r we have narrowed down the choices to two equations: $r = 1 + \cos \theta$ and $r = 1 + \sin \theta$.

Can you see which one must be the correct equation for Graph B? Try plotting a few points.

For equation 3: When $\theta = 0$ we have $r = 1 + \cos 0 = 2$. This matches the graph above.

For equation 4: When $\theta = 0$ we have $r = 1 + \sin 0 = 1$. This does not match the graph above.

This tells us that the equation must be 3: $r = 1 + \cos \theta$.



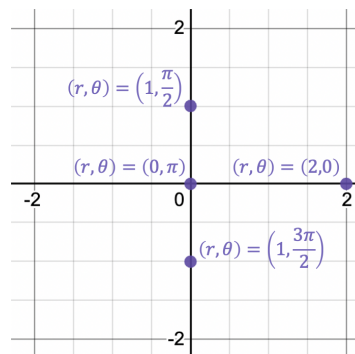
We will discuss how to sketch the graph of the polar equation $r = 1 + \cos \theta$ to see exactly why Graph B above matches this equation. You do not need to sketch this graph to complete the activity, but you may still want to spend some time thinking about why this is the correct graph.

For many of the eight equations, there are pairs (r, θ) with $r < 0$ that satisfy the equation. We discussed how to interpret negative r -coordinates on the last activity on Polar Coordinates.

Example 3: Sketch the graph of the polar equation $r = 1 + \cos(\theta)$.

Plot a few key points.

- When $\theta = 0$, $r = 2$.
- When $\theta = \frac{\pi}{2}$, $r = 1$.
- When $\theta = \pi$, $r = 0$.
- When $\theta = \frac{3\pi}{2}$, $r = 1$.
- When $\theta = 2\pi$, $r = 2$.

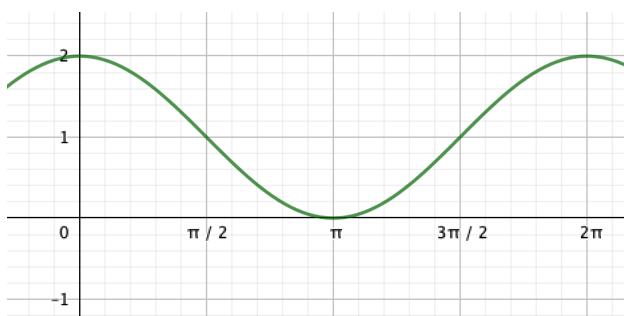


Think about the range of r .

Since $-1 \leq \cos \theta \leq 1$, we must have $0 \leq 1 + \cos \theta \leq 2$. This means all points on the graph must be at most 2 units from the origin.

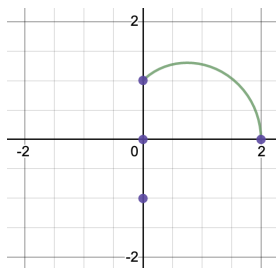
Think about how r changes as θ changes.

Can you describe what happens to r as θ ranges from 0 to 2π ? We sketch the graph of $y = 1 + \cos x$ drawn in the usual Cartesian plane. Can you see how to use this information to make the table?

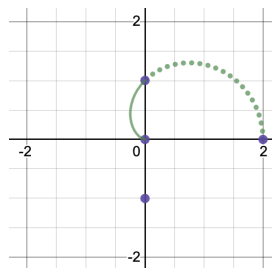


θ	$r = 1 + \cos(\theta)$	Polar Point
0	2	$(2, 0)$
0 to $\frac{\pi}{2}$	r decreases from 2 to 1	
$\frac{\pi}{2}$	1	$(1, \frac{\pi}{2})$
$\frac{\pi}{2}$ to π	r decreases from 1 to 0	
π	0	$(0, \pi)$
π to $\frac{3\pi}{2}$	r increases from 0 to 1	
$\frac{3\pi}{2}$	1	$(1, \frac{3\pi}{2})$
$\frac{3\pi}{2}$ to 2π	r increases from 1 to 2	
2π	2	$(2, 2\pi)$

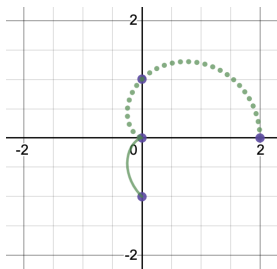
Draw a rough sketch of the curve



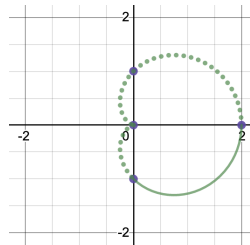
As θ increases from 0 to $\frac{\pi}{2}$, r decreases from 2 to 1. So we connect the polar points $(2, 0)$ and $(1, \frac{\pi}{2})$ through the first quadrant.



As θ increases from $\frac{\pi}{2}$ to π , r decreases from 1 to 0. So we connect the polar points $(1, \frac{\pi}{2})$ and $(0, \pi)$ through the second quadrant.



As θ increases from π to $\frac{3\pi}{2}$, r increases from 0 to 1. So we connect the polar points $(0, \pi)$ and $(1, \frac{3\pi}{2})$ through the third quadrant.



As θ increases from $\frac{3\pi}{2}$ to 2π , r increases from 1 to 2. So we connect the polar points $(1, \frac{3\pi}{2})$ and $(2, 2\pi)$ through the fourth quadrant.

Can you convince yourself that the sketch will take this curved shape? We used technology to plot many points in order to get an accurate curve. Since the function $\cos \theta$ repeats with period 2π , plotting points for more values of θ will just result in drawing this same curve over again!

Example 4: Consider the polar equation $r = 1 + 2 \sin \theta$.

Notice that there are values of θ for which the corresponding r is negative. For example, when $\theta = \frac{3\pi}{2}$, we have

$$r = 1 + 2 \sin \left(\frac{3\pi}{2} \right) = 1 + 2(-1) = -1$$

Recall from the Activity on Polar Coordinates that it is also possible to represent a Polar point using negative value of r !

Example 5: Consider the graph of the polar equation $r = 1 + 2 \sin \theta$.

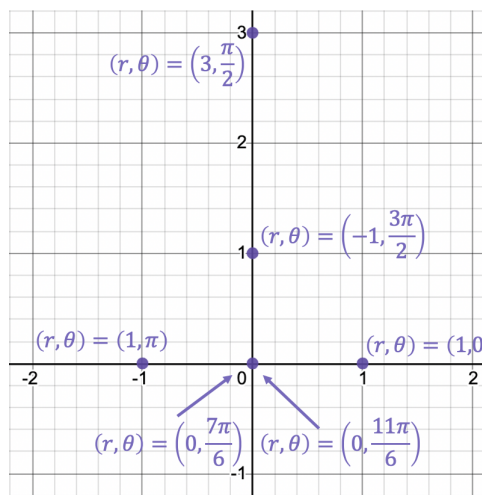
Note that it will be important to know where r changes from negative to positive. To find these places, we solve the equation $r = 1 + 2 \sin \theta = 0$. Two solutions are $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$.

Plot a few key points.

- When $\theta = 0$ (or $\theta = 2\pi$), $r = 1$.
- When $\theta = \frac{\pi}{2}$, $r = 3$.
- When $\theta = \pi$, $r = 1$.
- When $\theta = \frac{7\pi}{6}$, $r = 0$.
- When $\theta = \frac{3\pi}{2}$, $r = -1$.

Remember that this pair describes the same point as the pair $\theta = \frac{\pi}{2}$ and $r = 1$.

- When $\theta = \frac{11\pi}{6}$, $r = 0$.



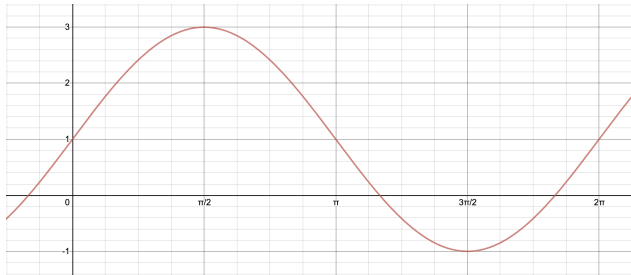
Think about the range of r .

Since $-1 \leq \sin \theta \leq 1$, we must have $-1 \leq 1 + 2 \sin \theta \leq 3$. Since the magnitude of r must be at most 3, we know that all points on the graph must lie at most 3 units away from the origin.



Think about how r changes as θ changes.

Can you describe what happens to r as θ ranges from 0 to 2π ?



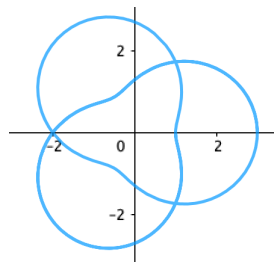
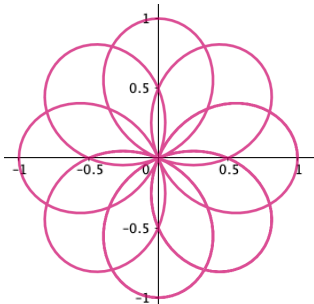
θ	$r = 1 + 2 \sin(\theta)$	Polar Point
0	1	$(1, 0)$
0 to $\frac{\pi}{2}$	r increases from 1 to 3	
$\frac{\pi}{2}$	3	$(3, \frac{\pi}{2})$
$\frac{\pi}{2}$ to π	r decreases from 3 to 1	
π	1	$(1, \pi)$
π to $\frac{7\pi}{6}$	r decreases from 1 to 0	
$\frac{7\pi}{6}$	0	$(0, \frac{7\pi}{6})$
$\frac{7\pi}{6}$ to $\frac{3\pi}{2}$	r decreases from 0 to -1	
$\frac{3\pi}{2}$	-1	$(-1, \frac{3\pi}{2})$
$\frac{3\pi}{2}$ to $\frac{11\pi}{6}$	r increases from -1 to 0	
$\frac{11\pi}{6}$	0	$(0, \frac{11\pi}{6})$
$\frac{11\pi}{6}$ to 2π	r increases from 0 to 1	
2π	1	$(1, 2\pi)$

It is not easy to see how to translate the complete information from the table into a sketch of the graph. It takes most people a lot of time to get comfortable sketching these curves when they involve negative values of r . Luckily, you do not need to sketch the *whole* curve in order to figure out which graph matches the equation $r = 1 + 2 \sin \theta$. If you can draw a few “pieces” of the graph for $r = 1 + 2 \sin \theta$ then you should be able to pick its graph out of the list. In fact, you might be able to pick out the correct graph by using *only* the key points considered in this example!

Try More:

You may also want to check out some of the free online graphing calculators for polar curves, like the ones offered by *WolframAlpha* or *Desmos* to verify your answers.

The graphs in the header of the first page of this activity each come from graphing one of the following polar equations. Which equation matches which graph and why?



$$r = 2 + \cos\left(\frac{3\theta}{2}\right)$$

$$r = \cos\left(\frac{4\theta}{3}\right)$$