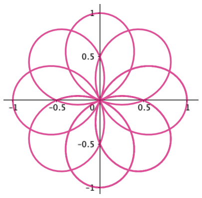
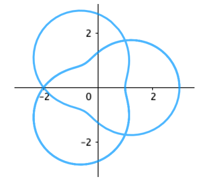
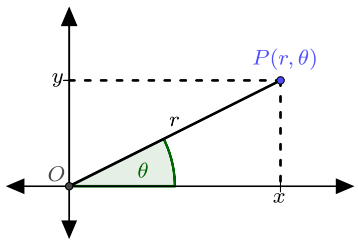


CEMC Math Circles - Grade 11/12

October 28 - November 3, 2020
Polar Curves - Solution





Relationships between Cartesian coordinates and polar coordinates of a point in the plane

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$r = \sqrt{x^2 + y^2}$$

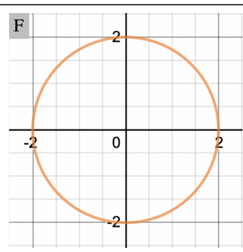
Let f be a function on the real numbers. The graph of the polar equation $r = f(\theta)$ consists of all points in the plane that have polar coordinates, (r, θ), that satisfy the relation $r = f(\theta)$.

Activity

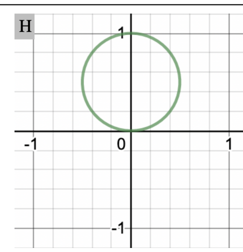
Consider the following polar equations and the graphs below. Exactly one of the graphs corresponds to each equation. Can you match each equation with its graph?

Answers (explanations provided on the pages that follow)

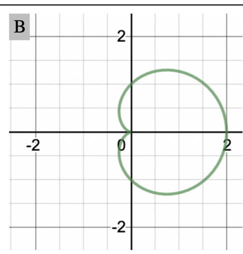
1. $r = 2$



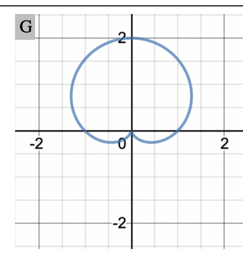
2. $r = \sin\theta$



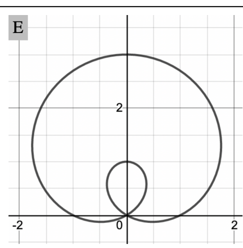
3. $r = 1 + \cos\theta$



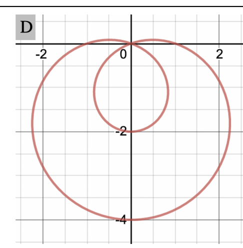
4. $r = 1 + \sin\theta$



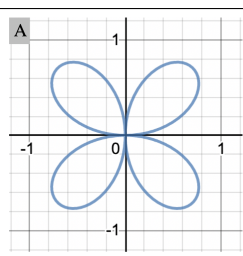
5. $r = 1 + 2\sin\theta$



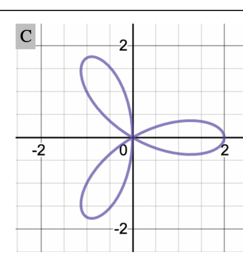
6. $r = 1 - 3\sin\theta$



7. $r = \sin(2\theta)$



8. $r = 2\cos(3\theta)$



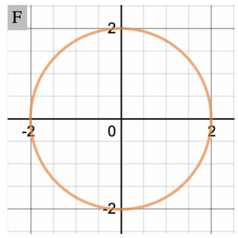
Step 1: Think about the range of r. *(This is only one of many possible first steps.)*

For the equations: Using the fact that $-1 \leq \sin\theta \leq 1$ and $-1 \leq \cos\theta \leq 1$, we can determine the range of r values for the polar functions $r = f(\theta)$.

For the graphs: We cannot determine the exact range of r values of the associated equation just by looking at the graph, but we can determine an upper bound on the *magnitude of* r from the graph.

1. $r = 2$

   Range: $2 \leq r \leq 2$



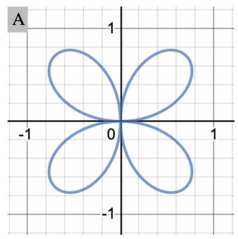
All points appear to be 2 units away from the origin.

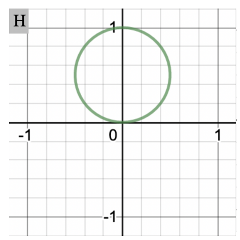
2. $r = \sin\theta$

   Range: $-1 \leq r \leq 1$

7. $r = \sin(2\theta)$

   Range of r: $-1 \leq r \leq 1$





The points that are farthest from the origin appear to be 1 unit away.

3. $r = 1 + \cos\theta$

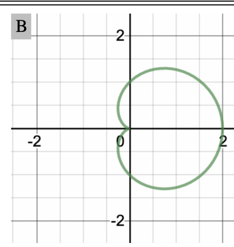
   Range of r: $0 \leq r \leq 2$

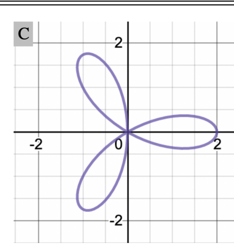
4. $r = 1 + \sin\theta$

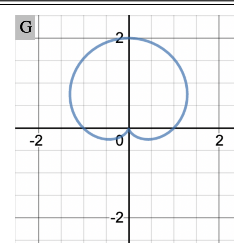
   Range of r: $0 \leq r \leq 2$

8. $r = 2\cos(3\theta)$

   Range of r: $-2 \leq r \leq 2$



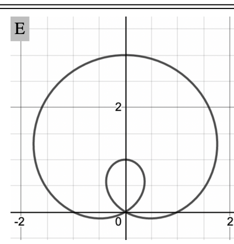




The points that are farthest from the origin appear to be 2 units away.

5. $r = 1 + 2\sin\theta$

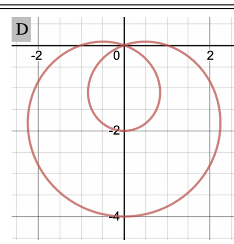
   Range of r: $-1 \leq r \leq 3$



The points that are farthest from the origin appear to be 3 units away.

6. $r = 1 - 3\sin\theta$

   Range of r: $-2 \leq r \leq 4$

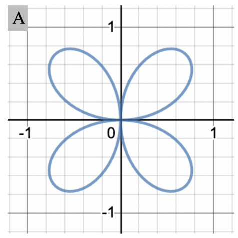


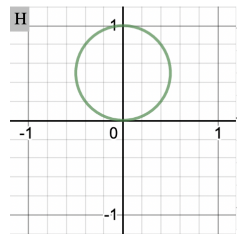
The points that are farthest from the origin appear to be 4 units away.

Step 2: Plot a few key points

Our work on the previous page matches equations 1, 5, and 6 with their graphs, and divides the remaining equations into two different groups as shown below. To determine which equation matches with which graph (within its group) we will think about plotting a few key points.

2. $r = \sin\theta$
7. $r = \sin(2\theta)$

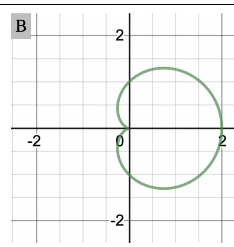


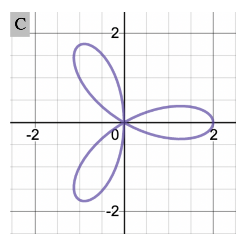


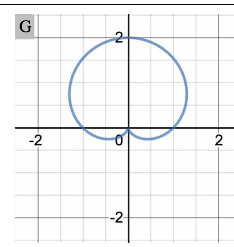
Consider the point in the plane with Cartesian coordinates $(x, y) = (0, 1)$. Notice that this point is on graph H above but not on graph A. One way to describe this point using polar coordinates is $(r, \theta) = (1, \frac{\pi}{2})$.

Since $1 = \sin\left(\frac{\pi}{2}\right)$, this point must be on the graph of equation 2: $r = \sin\theta$. This means equation 2 must be matched with graph H. It follows that equation 7 must be matched with graph A.

3. $r = 1 + \cos\theta$
4. $r = 1 + \sin\theta$
8. $r = 2\cos(3\theta)$







First, consider the point with Cartesian coordinates $(x, y) = (2, 0)$. Notice that this point is on graphs B and C above but not on graph G. One way to describe this point using polar coordinates is $(r, \theta) = (2, 0)$.

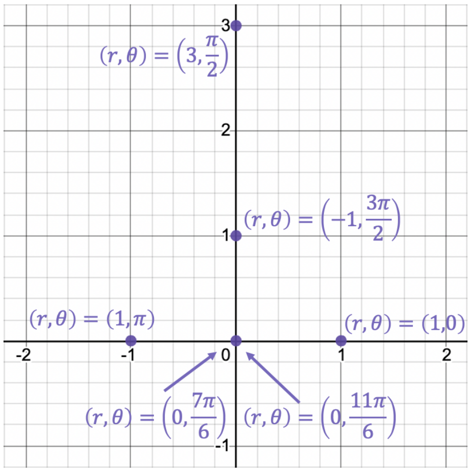
Since $2 = 1 + \cos(0)$ and $2 = 2\cos(3 \cdot 0)$, this point must be on the graphs of equation 3 ($r = 1 + \cos\theta$) and equation 8 ($r = 2\cos(3\theta)$). This means equations 3 and 8 must be matched with graphs B and C, in some order. It follows that equation 4 ($r = 1 + \sin\theta$) must be matched with graph G.

Now, consider the point with Cartesian coordinates $(x, y) = (0, 1)$ and polar coordinates $(r, \theta) = (1, \frac{\pi}{2})$. Notice that this point is on graph B but not on graph C.

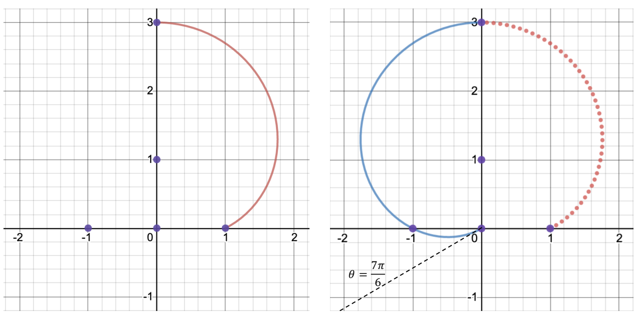
Since $1 = 1 + \cos\left(\frac{\pi}{2}\right)$, this point must be on the graph of equation 3: $r = 1 + \cos\theta$. This means equation 3 must be matched with graph B. It follows that equation 8 must be matched with graph C.

Notice that we have completed the matching activity without actually graphing any of the polar equations completely. We have just picked out certain characteristics of the equations and the graphs in order to find the right matches. We encourage you to follow the strategy outlined in the activity for how to sketch the graphs of the equations from scratch, and confirm the matchings that way as well. On the next page, we revisit the polar equation $r = 1 + 2\sin\theta$ that was discussed in the activity, and outline how to sketch its graph.

Using a few key points on the graph, and the table below, we sketch the graph of $r = 1 + 2\sin\theta$.



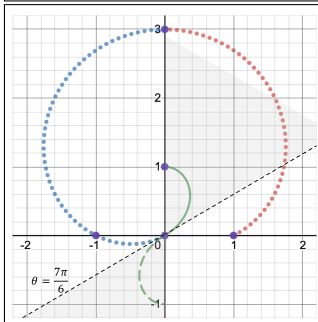
θ	$r = 1 + 2\sin(\theta)$	Polar Point
0	1	$(1,0)$
0 to $\frac{\pi}{2}$	r increases from 1 to 3	
$\frac{\pi}{2}$	3	$(3, \frac{\pi}{2})$
$\frac{\pi}{2}$ to π	r decreases from 3 to 1	
π	1	$(1,\pi)$
π to $\frac{7\pi}{6}$	r decreases from 1 to 0	
$\frac{7\pi}{6}$ to $\frac{3\pi}{2}$	r decreases from 0 to -1	
$\frac{3\pi}{2}$	-1	$(-1, \frac{3\pi}{2})$
$\frac{3\pi}{2}$ to $\frac{11\pi}{6}$	r increases from -1 to 0	
$\frac{11\pi}{6}$ to 2π	r increases from 0 to 1	
2π	1	$(1,2\pi)$



As θ increases from 0 to $\frac{\pi}{2}$, r increases continuously from 1 to 3. So we connect the polar points $(1,0)$ and $(3, \frac{\pi}{2})$ through the first quadrant as shown in the leftmost image.

As θ increases from $\frac{\pi}{2}$ to π, r decreases from 3 to 1. So we connect polar points $(3, \frac{\pi}{2})$ and $(1,\pi)$ through the second quadrant.

As θ increases from π to $\frac{7\pi}{6}$, r decreases from 1 to 0. So we connect polar points $(1,\pi)$ and $(0, \frac{7\pi}{6})$ through the third quadrant as shown in the rightmost image.

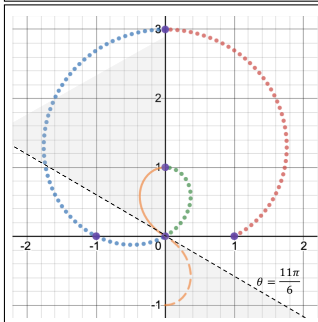


As θ increases from $\frac{7\pi}{6}$ to $\frac{3\pi}{2}$, r decreases from 0 to -1.

Since the point $(-1, \frac{3\pi}{2})$ is equivalent to the point $(1, \frac{\pi}{2})$, we know we are connecting the points $(0, \frac{7\pi}{6})$ and $(1, \frac{\pi}{2})$. But how?

Since r is negative between $\theta = \frac{7\pi}{6}$ and $\theta = \frac{3\pi}{2}$, the points we plot for this interval of θ will not actually lie in the third quadrant; they will lie in the first quadrant as indicated in the image. The magnitude of r tells us how far to plot the points from the origin; the negative sign attached to r tells us to plot the points on the "other side of the origin".

We connect the points $(0, \frac{7\pi}{6})$ and $(1, \frac{\pi}{2})$ through the first quadrant as shown.

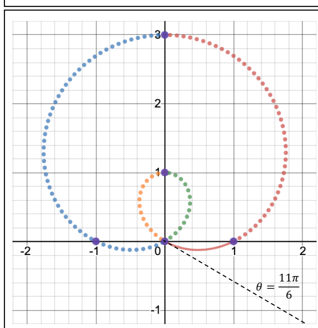


As θ increases from $\frac{3\pi}{2}$ to $\frac{11\pi}{6}$, r increases from -1 to 0.

Since the point $(-1, \frac{3\pi}{2})$ is equivalent to the point $(1, \frac{\pi}{2})$, we know we are connecting the points $(1, \frac{\pi}{2})$ and $(0, \frac{11\pi}{6})$. But how?

Since r is negative between $\theta = \frac{3\pi}{2}$ and $\theta = \frac{11\pi}{6}$, the points we plot for this interval of θ will not actually lie in the fourth quadrant; they will lie in the second quadrant as indicated in the image.

We connect the points $(1, \frac{\pi}{2})$ and $(0, \frac{11\pi}{6})$ through the second quadrant as shown.



Finally, we finish the sketch for θ between $\frac{11\pi}{6}$ and 2π, where r increases from 0 to 1.

Since the function $\cos\theta$ repeats with period 2π, plotting points for more values of θ will just result in drawing this same curve over again.

The completed graph is shown on the right.