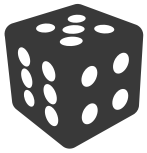
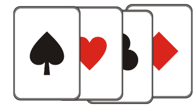


# CEMC Math Circles - Grade 11/12

## November 11 - 17, 2020
## Mathematical Games



This week, we're going to look at *mathematical games*. They seem a bit abstract, but the ideas have far-reaching applications to game theory, economics, and life! This topic is very accessible, and doesn't require any background. There will be **three** main examples– try to work through one per day. Play some of these games with a friend or a sibling! If you have fun solving these problems, there are *lots* of similar (and harder) problems at the end of the worksheet.

**Disclaimer:** Don't google answers to these problems! Solutions will be posted later in the week– it's much more fun to work them out yourself.

We will study **combinatorial games**. These games are characterized by the following properties:

- Two players **take turns** (most games!);
- **No luck or chance** is involved (this rules out many card games); and
- Players have **perfect information**: in other words, any information about the game is available to both players (this also rules out many card games).

If we say that the game is **impartial**, we mean the following property:

- **Either player may make any move** (this rules out chess or checkers);

**Question.** What games do **you** know that satisfy the criteria above? (Think about some common games: Checkers, Chess, Go, Tic Tac Toe, etc. Do any of these satisfy all the above criteria?)

What kinds of games satisfy these properties in general? As it turns out, quite a few! As a warm-up, we'll look at one from the popular TV show *Survivor*.

**Warm-up Activity:** Watch this video on YouTube describing the game **21 Flags**.

The rules of **21 Flags** are the following:

- There are 21 flags.
- Two players take turns removing either 1, 2, or 3 flags.
- The player who takes the last flag wins.

**Questions.** Answer the following:

(a) Check that this is actually an *impartial combinatorial game* as described above.
(b) Actually play this game with a friend, sibling, or parent, or online here. Do you notice any patterns?
(c) Can you always win this game? Does it matter if you start first or second?
(d) If so, describe how to do this, i.e., a *winning strategy* for **21 Flags**.

While playing **21 Flags**, you may have noticed that who goes first or second *does* matter eventually. In an impartial combinatorial game, **there are no ties**, since someone must win. If one of the players can guarantee they will win with a certain strategy, we will call this a **winning strategy**.

> A player of an impartial combinatorial game has a **winning strategy** if **they can guarantee they will win**. One of the two players *always* has a winning strategy. (Why?)

We're going to look at a few games, and determine winning strategies for each. If you liked **21 Flags**, check out the extra problems in the Warm-up section at the end of this worksheet.

The first two games fall under the umbrella of "diagonal" games (we will see why soon). For each game, read the instructions, and play it to get a sense of the strategy involved.

**The Calendar Game:**

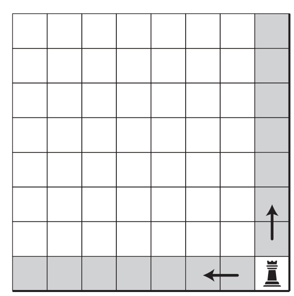

Players take turns writing down dates. The first player must begin by writing down January 1. After this, the next player takes the previous date and may increase either the month or the day, but *not both*. For example, the second player could choose January 12, or May 1, but not Feburary 2. The player who writes down December 31 wins.

**Hint:** Try to work backwards. *This can be played by simply writing down each date.*

**A Rook on a Chessboard:**

A rook is placed on the bottom right square of an $8 \times 8$ chessboard. On each players turn, they move it any number of spaces to the left, or any number of spaces up (never to the right or down). The player who moves the rook to the top left square wins.

*This can be played on a chessboard if you have one at home. If not, there is a grid provided at the end of this worksheet to print out.*



> **Activity #1:** Play **The Calendar Game** and **A Rook on a Chessboard** and see if you notice any patterns.
>
> **Questions.** Answer the following:
> (a) Check that these are actually *impartial combinatorial games.*
> (b) Which player (first or second) has a winning strategy for each game? What is the strategy?
> (c) What makes these games similar? How are the strategies related?

If you liked these two games, check out the extra problems in the section labeled Activity #1 at the end of this worksheet.

You may have noticed something special about certain dates in **The Calendar Game** if you followed the hint. Sometimes, there are positions, or states of the game (in this case, dates) from which you can *always win*. In other words, if it's your turn, you can make a series of moves so that you will win *no matter what your opponent does*. This is a really useful idea!

> A position in a game is called a **winning position** if you can guarantee you will win from this position. In other words, if it's your turn, you can eventually win **no matter what your opponent does.** Any position which is *not* a winning position is called a **losing position**.

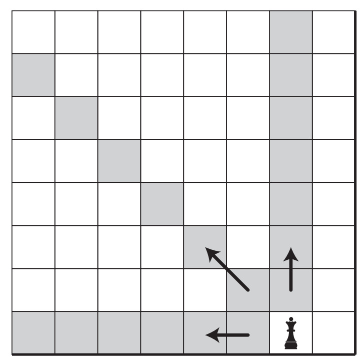
**Warning:** It's *not* true that you can't win from a losing position! You might be able to win, but it *depends* on what your opponent does.

Sometimes, a winning strategy isn't very obvious, and we have to work backwards. What we're really doing is recording all of the winning positions! With this in mind, we'll look at another game.

### The Left Handed Queen:

A queen is placed near the bottom right square of an $8 \times 8$ chessboard. On each players turn, they can move it any number of spaces to the left, diagonally up and to the left, or up. The player who moves the queen to the top left square wins.

*This can be played on a chessboard if you have one at home. If not, there is a grid provided at the end of this worksheet to print out.*



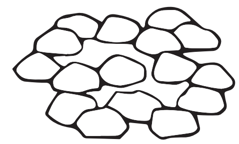
> **Activity #2:** Play **The Left Handed Queen** and see if you notice any patterns.
>
> **Questions.** Answer the following:
> (a) Check that this is actually an *impartial combinatorial games.*
> (b) Label all the winning and losing positions on a chessboard. **Hint:** Work backwards.
> (b) Which player (first or second) has a winning strategy? What is the strategy? What if the queen starts somewhere else?
> (d) What if the chessboard was very large (say, $100 \times 100$), and the queen were placed somewhere near the bottom? How would you determine whether this was a winning or losing position?

If you liked this game, check out the extra problems in the section labeled Activity #2 at the end of this worksheet.

The last game we'll look at is a very classic example of an impartial combinatorial game called **Nim**. There are many articles about this game online, but try to think about the questions before you look up any hints! It's much more fun if you can figure it out for yourself. All of these games are generally called "subtraction" games. As a warm up, we'll look at a simple version of **Nim**.

**Easy Nim:**

The game begins with two piles of stones. One has 5 and the other has 7. On their turn, each player may take *any* number of stones from one pile. The player who takes the last stone loses.



*The easiest way to play this game is with some coins or similar small objects.*

**Question.** Have we seen this game before? Which of the previous games is most similar to this one?

**Nim:**

The game begins with five piles of stones. There are $1, 2, 3, 4$, and 5 stones in each pile (to save space, we might write this as $1 \oplus 2 \oplus 3 \oplus 4 \oplus 5$). On their turn, each player may take *any* number of stones from one pile. The player who takes the last stone loses.

You can play this game online here, but don't click on the explanation! *There are more general versions, but this one is already hard enough. Check out the extra problems if you want a challenge.*

**Activity #3:** Play **Nim** (online or with a partner) and see if you notice any patterns.

**Questions.**
- (a) Check that **Nim** is actually an *impartial combinatorial game.*
- (b) Can you write down the winning and losing positions for **Easy Nim** and **Nim**? How many are there?
- (b) In each game, which player (first or second) has a winning strategy? How do you know?

Figuring out the winning strategy for **Nim** is tricky if you've never seen it before (and if you don't look it up). Here are some steps to get you started.

- (a) Is $1 \oplus 2$ a winning or losing position? What about $2 \oplus 3$?
- (b) Suppose that from some position, there is a move to $1 \oplus 2$ (for instance, $1 \oplus 2 \oplus 4$ is such a position). Can you tell whether this is a winning or losing position?

If there are more piles with an arbitrary number of stones, can you describe the winning strategy?

If you liked **Nim**, check out the extra problems in the section labeled Activity #3 at the end of this worksheet.

**Challenge Problems:**

If you liked those games, check out these ones! They're organized by how they relate to the activities in this worksheet.

**Warm-up:**

- **Generalized 21 Flags**: There are $n$ flags. On their turn, a player can remove 1, 2, 3,..., or $k$ flags. The player who takes the last flag wins.
- **Even More 21 Flags:** Here are two more versions of **21 Flags** to think about.
  - There are 21 flags. On their turn, a player can remove 1, 2, or 4 flags. The player who takes the last flag wins.
  - There are $n$ flags. On their turn, a player can remove 1, 3, or 5 flags. The player who takes the last flag wins.
- **The Subtracting Game:** Similar to **21 Flags**, this game starts with 44 flags. Player 1 can remove any number of flags, but must leave at least one. Thereafter, players may remove *at most* as many as the previous player did. The player who takes the last flag wins.
- **Yet Another Flag Game:** Like **The Subtracting Game**, this game starts with 44 flags. Player 1 can remove any number of flags, but must leave at least one. Thereafter, players may remove *up to twice as many* as the previous player did. The player who takes the last flag wins.

**Activity #1:**

- **Erase from 13**: A chalkboard has the numbers $1, 2, 3, \ldots, 13$ written on it. Two players take turns erasing a number from the board, until two numbers remain. The first player wins if the sum of the last two numbers is a multiple of 3. Otherwise, the second player wins. What if we start with the numbers $1, 2, 3, \ldots, 2020$?
- **Bishop:** This game is played on an $8 \times 8$ chessboard. Players take turns placing a bishop anywhere on the chessboard so that it cannot be captured by the other bishops on the board. In other words, no two bishops are on the same diagonal line. The player who can no longer may a move loses. Show that the second player can always win.

**Activity #2:**

- **A Knight on a Chessboard Game?** The game begins with a knight placed on an $8 \times 8$ chessboard. Players take turns moving the knight in the usual L-shaped moves. The player who moves it to the top left corner wins.
  - Why is this *not* a combinatorial game?
- **A Real Knight on a Chessboard Game** There is a way to make the above game into a real combinatorial game. The first player *chooses* a starting position for a knight anywhere on an $8 \times 8$ chessboard. Afterwards, players take turns moving the knight to a position *that*

*has not been visited before* (in other words, no repeated positions are allowed). The player who has no more available moves loses.

- Why is this a combinatorial game?
- This is a hard problem! Show that the second player has a winning strategy if the board has size $2 \times 4$. If you can do this, show that the second player has a winning strategy if the game is played on a $4 \times 4$ or $5 \times 5$ chessboard.
- Can a knight placed on a chessboard visit every square exactly once?
- If you can do this, prove that the second player has a winning strategy on a regular $8 \times 8$ chessboard.

**Activity #3:**

- **Circular Nim:** The game begins with 10 stones placed in a circle. Players take turns removing up to three *consecutive* stones. The player who takes the last stone wins.
- **Fibonacci Nim:** This game starts with 44 coins in a pile. Player 1 can remove any number of coins, but must leave at least one. Thereafter, players may remove *up to twice as many* as the previous player did. The player who takes the last coin wins. **Hint:** This game is named after the *Fibonacci sequence* $\{1, 2, 3, 5, 8, 13, 21, ...\}$. What is this sequence? What does it have to do with this game?

**A chessboard for some of the games:**