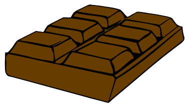
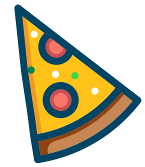


CEMC Math Circles - Grade 11/12

November 18 - 24, 2020
Partitioning Puzzles - Solutions



Here are the solutions to this weeks problems. There is a description of each problem, followed by a discussion about the solution.

Problem 1. Once a day, you break a large 4×8 rectangular chocolate bar into its smaller 1×1 pieces, to distribute to your class of 32 students. After some time, you notice that *no matter how* you break up the chocolate bar, it always requires the *same* number of breaks. How many does it take? Why?

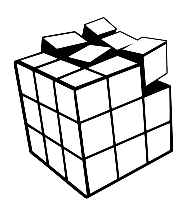
Remarks: You can only break the chocolate along its grid lines, and you can only break a *single* piece at a time (the way one might actually break up a chocolate bar). If you can do this problem, consider a chocolate bar of size $m \times n$. Now how many does it take?

Solution. It will always take 31 breaks to break the chocolate bar into its 1×1 squares. You can think about all of the possible ways to break it up (there are many), but working backwards is much easier. Each time you make a break, you will have *exactly one* more piece (maybe not of size 1×1) than you did before. Working backwards, this means you will always make 31 breaks (one less than the total number of squares). If the chocolate bar had size $m \times n$, you would always perform $mn - 1$ breaks.

Try this with some small chocolate bars to see it in action! □

Problem 2. You begin with a cube made from 27 smaller cubes glued together (arranged just like a Rubik's cube, although this has nothing to do with this problem). You would like to cut it into its 27 component cubes with a large knife– what is the minimum number of cuts required to do so? How do you know?

Remarks: This is not a trick question. The knife only cuts in a straight line; it does not bend, and you cannot move the pieces while you are making a cut. You may re-arrange, stack, or assemble the pieces however you would like in between each cut, so that you cut more than one piece at the same time. While reading this remark, you should try to convince yourself that you need at *least* 4 cuts.



Solution. This is a slightly tricky question. First, we note that we can certainly separate the cube into 27 pieces using 6 cuts: just cut twice along 3 different faces without every moving any of the pieces. The question is: if we're allowed to re-assemble the pieces in between each cut, can we do better?

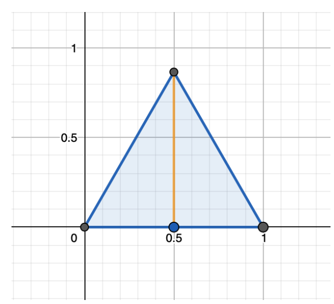
The answer is no! Think about the cube at the very centre. No matter how you re-assemble the pieces, it will take always 6 cuts to separate it from all of its neighbours, since those cuts can't be performed at the same time. Thus, we'll always need at least 6 cuts.

Perhaps surprisingly, a $4 \times 4 \times 4$ Rubik's cube can also be cut into its 64 pieces in only 6 moves (figure out how). If you thought this was easy: how many cuts does it take to cut apart an $n \times n \times n$ cube? $\square$

Problem 3. You have a single piece of pizza, in the shape of a (perfect) equilateral triangle with side length 1 unit. You and your friend would like to divide it equally: what is the *length* of the shortest curve which cuts the triangle into two pieces of equal area?

Remarks: This is not a trick question. The curve does not have to be a straight line, nor does it have to start or end at any of the corners. The *median* of the triangle (in red on the right) is one curve that works; you should convince yourself that the length of this curve is $\sqrt{3}/2$. Can you do better than this?

Solution. Of the three problems, this one is the hardest. We'll start by setting some notation: we will call the equilateral triangle T. Note that all of the interior angles of T are all equal to 60°. In radians, this is simply $\pi/3$. If we want, we can plot this triangle in the plane with two vertices at $(0, 0)$ and $(1, 0)$ (like all of the pictures below).



We can do some trigonometry to figure out the length of the median (illustrated in orange on the right): it is the opposite side of a triangle with angle 60° and hypotenuse equal to one, so has length $\sin(60°) = \sqrt{3}/2 \approx 0.866$. Even better, the area of a triangle is equal to (base $\times$ height)/2, so using the median (the height), we find that the area of T is $(1 \cdot \sqrt{3}/2)/2 = \sqrt{3}/4 \approx 0.433$.

Thus, the median is one possible curve which splits T into two regions of equal area, and has length $\sqrt{3}/2$. The question is: can we do any better?

There are essentially two possibilities: the curve has endpoints on the "same side," or on "two different sides." The first case is easier, and maybe something you discounted right away without giving a rigorous justification. To deal with this, we'll use the following fact, which will appear later:

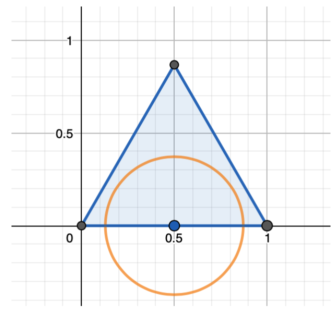
A circle is the shortest curve among all curves enclosing a fixed area.

Take a moment to parse what this means. In other words, if we want to enclose a region of some fixed area, the shortest curve which does so is a circle. This seems obvious when you think about it, but proving it rigorously is surprisingly hard. If you'd like to know how, ask me!

In particular, if we want to enclose the most area with the shortest curve that starts and ends on one side of T, it had better be a semicircle! If we use a semi circle (centered at $(1/2, 0)$), we can figure out the required radius r fairly easily. The area enclosed by the semicircle is $\pi r^2/2$, but on the other hand this has to be $\sqrt{3}/8$ (half of the area of T). Rearranging, we find that:

$$\frac{\pi r^2}{2} = \frac{\sqrt{3}}{8} \implies r = \sqrt{\frac{\sqrt{3}}{4\pi}} \approx 0.371$$

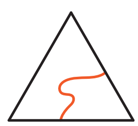
If we plot the circle with this radius centred at $(1/2, 0)$, we get the picture on the right. Moreover, the length l of this curve is exactly half of the circumference, so we get:

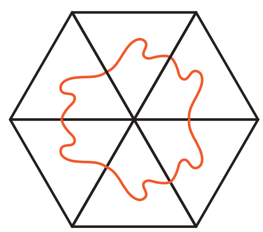


$$l = \frac{\pi \cdot 2r}{2} = \pi r = \pi\sqrt{\frac{\sqrt{3}}{4\pi}} \approx 1.166$$

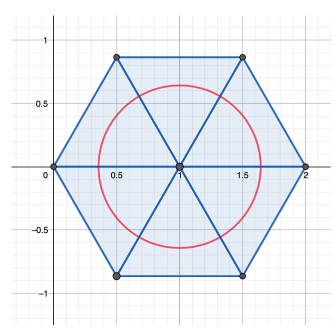
This is way worse than our original candidate! We conclude that if the curve we choose has its endpoints on the same side, the shortest it could be is 1.166.

What about the second case? This time we'll need a trick, since our useful fact about circles doesn't really apply anymore. Suppose that we have a curve that starts and ends on different sides of T. Then we can take six copies of T (and our curve) to form a hexagon, and a closed curve! You might worry that the endpoints won't match up; you can check that if we're careful about how we do this, they will.





Great! Now we do have a closed curve, and we know that it must have the shortest length among all curves enclosing exactly half the area of the hexagon. Therefore, it must be a circle! Since the area of the hexagon is six times the area of T, we can figure out what the radius R must be in this case. We find that:

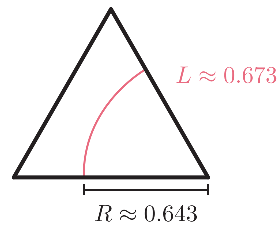


$$\pi R^2 = \frac{1}{2}(6 \cdot \sqrt{3}/4) \implies R = \sqrt{\frac{3\sqrt{3}}{4\pi}} \approx 0.643$$

Even better, the length L of the curve (the portion of the circle in T) is exactly $1/6$ of the circumference of this circle, so we get:

$$L = \frac{1}{6} \cdot \pi(2R) = \frac{\pi}{3} \cdot \sqrt{\frac{3\sqrt{3}}{4\pi}} \approx 0.673$$

This is better than our estimate using the median! We conclude that if the curve we chose has endpoints on two different sides, this is the best we can do. In particular, this is the length of the shortest curve which cuts an equilateral triangle into two regions of equal area. □



Other Remarks: If you'd like to know how to prove that circles have the least perimeter among all shapes with a fixed area, let me know! If you could do this problem or thought this solution was cool, try doing this for other regular polygons. What if T was a square? What if T was a pentagon?