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Grade 6 Math Circles

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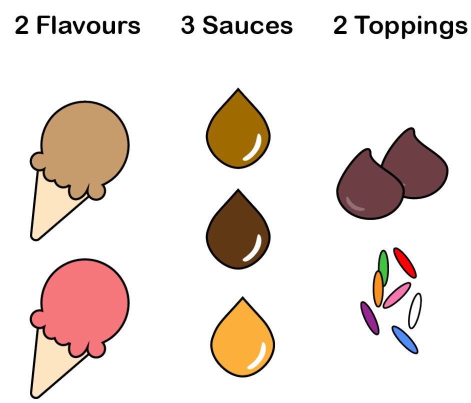
Counting Part I

Today we will be learning how to count the number of possible outcomes for specific actions and events. This type of mathematical counting is the foundation of an entire branch of math called **combinatorics**. How is this different from counting "1, 2, 3,. . . "? Let's find out!

How Many Choices?

Example: Ice Cream, Ice Cream!

Suppose you are making an ice cream sundae. There are 2 flavours, 3 sauces, and 2 toppings that you can choose from. You are to select one of each. How many different sundaes could you make?



Ice Cream, Ice Cream! Video Solution:
https://youtu.be/8Y8GbQ0lpZc

What if I had 8 flavours, 6 sauces, and 4 toppings? We can solve this by writing down each option like we did in the video above. You may notice that with more options, using this method will take a very long time! How can we calculate this more effeciently?

Fundamental Counting Principle:
If you have to make Choice A **AND** Choice B, and there are m options for Choice A and n options for Choice B, then the total number of different ways you can make Choice A and Choice B is $m \times n$.

This principle applies in the same way if more choices have to made. We simply have to multiply the options for each choice.

When you see "**AND**" in a counting problem, it is a hint that you have to use this principle! Let's apply this rule to the Ice Cream, Ice Cream! example.

Example: Ice Cream, Ice Cream! continued

First let's count the total number of choices and options that we have.

- Since we can choose an ice cream flavour, a sauce, and a topping for each sundae, we have **3** total choices.
- For options, there are **2** options for flavour, **3** options for sauce, and **2** options for topping. Now using the Fundamental Couting Principle we have:

$$\text{total number of different sundaes} = \underset{\text{flavours}}{2} \times \underset{\text{sauces}}{3} \times \underset{\text{toppings}}{2} = 12$$

This is the same answer that we got using the earlier method!

We can also use this approach to solve the total number of different sundaes given 8 flavours, 6 sauces, and 4 toppings. We still have **3** total choices. For options, there are **8** options for flavour, **6** options for sauce, and **4** options for topping. Now using the Fundamental Couting Principle we have:

$$\text{total number of different sundaes} = \underset{\text{flavours}}{8} \times \underset{\text{sauces}}{6} \times \underset{\text{toppings}}{4} = 192$$

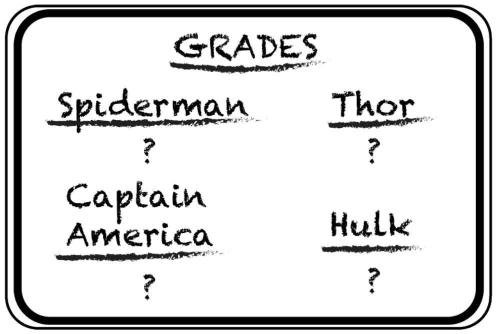
Imagine having to count 192 different sundaes like we did in the video. The Fundamental Couting Principle made counting this much simpler!

Repetition

In this lesson, we will look at two types of counting problems where **repetition is allowed**, and **repetititon is not allowed**. What does "repetition" mean in this case? Let's take a look at some examples.

Example: The Best Hero

Iron Man gives Spiderman, Thor, Captain America, and Hulk a task each to test their fighting abilites. Each hero could get one of 6 grades: **Outstanding, Exceeds Expectations, Acceptable, Poor, Dreadful, or Atrocious.** Iron man will then call a meeting with the Avengers and write their grades on the board. Assume it doesn't matter which heroes' grade he lists first, second, third, or fourth on the board.



1. Can Iron Man repeat any grades (i.e. give the same grade to more than one hero)?

2. How many different ways can the heroes be graded? Use Fundamental Counting Principle (FCP).

Solution:

1. Yes! - Since the same grade can be given to more than 1 hero, he can repeat grades. This is an example of a case where **repetition is allowed**. When repeptiton is not allowed the number of options for each choice is the same.

2. Using the FCP there are 4 choices Iron Man needs to make. He has to choose each of the **4 heroes' grade**. There are 6 options for each grade, **Outstanding, Exceeds Expectations, Acceptable, Poor, Dreadful, or Atrocious**. Thus:

$$\text{the total number of different ways to grade} = 6 \times 6 \times 6 \times 6 = 1296$$

Example: Bake a Cake

Suppose you are making a four-layer cake and have four flavours of cake to choose from: vanilla, chocolate, strawberry, and red velvet. Each layer of this cake must be a different flavour.

1. Is repetition allowed (i.e. using the same flavour more than once)?
2. How many different four-layer cakes can you make?

Bake a Cake Video Solution: *https://youtu.be/A3-BFZu-KFk*

In this example **repetition was not allowed**. When repetition is not allowed we have to reduce the number of available options each time.

Factorials

Whenever we have a list or group of n things and we need to figure out how many different ways they can be put in with **no repetition**, we use the same type of logic that we just applied to the previous example. This will always result in the answer being $n \times (n-1) \times (n-2) \times (n-3) \times ... \times 2 \times 1$. In our previous example our answer was $4 \times 3 \times 2 \times 1$. In other words, you are using the Fundamental Counting Principle to multiply the options for each choice in the order starting at n and decreasing by 1 every time a choice in the order is filled.

Factorial notation is used to write this operation:

$$n! = n \times (n-1) \times (n-2) \times ... \times 2 \times 1$$

$n!$ is read as "n-factorial".

Examples:

1. $0!$ is a special case. We can say that $0! = 1$
2. $1! = 1$
3. $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

4. $11! = 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 39916800$

Example: What's the Password?

Harry and Ron are trying to figure out the password for a secret passage behind the mirror on the fourth floor. They know the 7-digit long password uses the whole numbers from 1 to 7 but they do not know what order.

1. How many different passwords could there be if the digits **can repeat**?
2. How many different passwords could there be if the digits **cannot repeat**?

Solution:

1. There are 7 **choices** since the password is 7-digits long. Since repetition is allowed (i.e. we **can repeat** digits), there are 7 **options** for each digit (the numbers 1 to 7). Then,

$$\begin{aligned} \text{total number of different passwords} &= \underset{\text{digit 1}}{7} \times \underset{\text{digit 2}}{7} \times \underset{\text{digit 3}}{7} \times \underset{\text{digit 4}}{7} \times \underset{\text{digit 5}}{7} \times \underset{\text{digit 6}}{7} \\ &\times \underset{\text{digit 7}}{7} \\ &= 823,543 \end{aligned}$$

2. Since repetition is not allowed (i.e. the digits **cannot repeat**) we can only use each number once. For the first digit of our password we have 7 **options** to choose from. The digits: 1, 2, 3, 4, 5, 6, or 7.

$$\text{total number of different passwords} = \underset{\text{digit 1}}{7} \times \underset{\text{digit 2}}{} \times \underset{\text{digit 3}}{} \times \underset{\text{digit 4}}{} \times \underset{\text{digit 5}}{} \times \underset{\text{digit 6}}{} \times \underset{\text{digit 7}}{}$$

For the second digit of our password we have 6 **options** to choose from, since we chose a number from 1 to 7 for our first digit.

$$\text{total number of different passwords} = \underset{\text{digit 1}}{7} \times \underset{\text{digit 2}}{6} \times \underset{\text{digit 3}}{} \times \underset{\text{digit 4}}{} \times \underset{\text{digit 5}}{} \times \underset{\text{digit 6}}{} \times \underset{\text{digit 7}}{}$$

For the third digit of our password we have 5 **options** to choose from, since we chose

2 numbers from 1 to 7 for our first and second digit.

$$\text{total number of different passwords} = \underset{\text{digit 1}}{7} \times \underset{\text{digit 2}}{6} \times \underset{\text{digit 3}}{5} \times \underset{\text{digit 4}}{} \times \underset{\text{digit 5}}{} \times \underset{\text{digit 6}}{} \times \underset{\text{digit 7}}{}$$

We notice that this is just the factorial of 7 since there is **no repeption**. Thus,

$$\begin{aligned}\text{total number of different passwords} &= 7! \\ &= \underset{\text{digit 1}}{7} \times \underset{\text{digit 2}}{6} \times \underset{\text{digit 3}}{5} \times \underset{\text{digit 4}}{4} \times \underset{\text{digit 5}}{3} \times \underset{\text{digit 6}}{2} \\ &\times \underset{\text{digit 7}}{1} \\ &= 5040\end{aligned}$$

Permutations

What if we had n options in total to choose from but we only needed to order k of them? For example, suppose we wanted to know how many different 4-digit long passwords there are using the whole numbers from 1 to 7, without repeating digits. We have 7 (n) options but only need to order 4 (k) of them.

Permutations are a way of counting in this type of situation when there is **no repetition**.

$$_nP_k = \frac{n!}{(n-k)!}$$

This is read as "n permute k" and counts how many ways we can order k objects from a total of n objects

Example: Quidditch Tryouts

The National Quidditch team is holding tryouts for one Keeper, one Seeker, and one Chaser. Dimitar, Viktor, Georgi, Boris, Bogomil, Nikola, and Stoyanka all try out.

1. How many players total do we have to choose from and how many do we need to choose?
2. Is repetition allowed in our choices? Why?
3. Use the Fundamental Counting Principle to find how many ways the three positions

can be filled.

4. Now use the permutation formula to find how many ways the three positions can be filled.

Quidditch Tryouts Video Solution: *https://youtu.be/XUaake50tz0*

Example: How many different 3-letter words can you make using the letters from the word FLOWER with no repetition (i.e. you can only use each letter once in the 3-letter word)? They don't have to be valid words in English.

Solution:

In this example, **repetition is not allowed** since we can use each letter only once. This means we can use the permutation formula to find the number of different 3-letter words that we can make. Here $n = 6$ since we have 6 letters in the word FLOWER, and $k = 3$ since we need to choose/order 3-letters. Then,

$$_nP_k = \frac{n!}{(n-k)!} = {}_6P_3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = 6 \times 5 \times 4 = 120$$

Using the word FLOWER we can make 120 different 3-letter words.

Example: How many different 7-character licence plates can be made if the first 4 characters must be digits from 0 to 9 with no repetition, and the last 3 characters must be letters of the alphabet with repetition? (**Hint:** Remember the Fundamental Counting Principle)

Solution:

- For the first 4 characters since **repetition is not allowed**, using digits from 0-9, there are **10** options for the first digit, **9** options for the second digit, **8** options for the third digit, and **7** options for the fourth digit.

- For the last 3 characters, we have **26** options (A to Z) for each letter since **repetition is allowed.** Thus,

$$\begin{aligned}\text{total number of different license plates} &= \underset{\text{digit 1}}{10} \times \underset{\text{digit 2}}{9} \times \underset{\text{digit 3}}{8} \times \underset{\text{digit 4}}{7} \times \underset{\text{letter 1}}{26} \times \underset{\text{letter 2}}{26} \\ &\times \underset{\text{letter 3}}{26} \\ &= 88583040\end{aligned}$$