UNIVERSITY OF WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

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Counting Part II

By now you should be a pro at using the Fundamental Counting Principle and solving basic problems using the Permutation formula. In this lesson, we will build on our knowledge of Permutations and introduce another method of counting called **Combinations!**

Permutations with Repeats

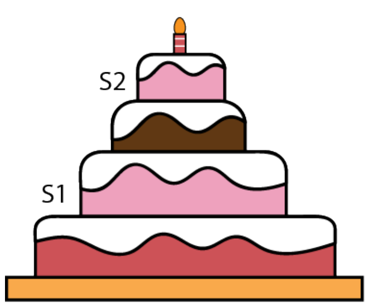
Example: Bake a Cake continued

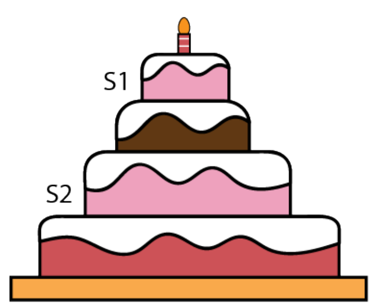
Suppose you are making a four-layer cake and have four flavours of cake to choose from: **strawberry, chocolate, strawberry, and red velvet**. Each layer of this cake must be a different flavour. How many different four-layer cakes can you make? Previously we saw:

$$\text{total number of different cakes} = 4! = 4 \times 3 \times 2 \times 1 = 24$$

Is this correct? You may have noticed that we list the flavour strawberry twice. Does this change how we previously solved this problem?

Yes! - Let's take a further look. In this example, each four-layer cake will have two layers that are strawberry. To help us understand this example let's call one of the strawberry layers **S1** and the second strawberry layer **S2**. Consider the following cakes:





If you look at the two cakes, there are two layers of strawberry in each. Both strawberry layers are identical: we can put S1 as the second layer in our cake, and S2 as the fourth layer, or we can put S2 as the second layer in our cake, and S1 as the fourth layer. It **does not matter** which order we put these strawberry layers in since they give us the exact same cake.

This means that for every four-layer cake we make, we can simply swap the two flavours (S1 and S2) and get the exact same cake again. Because of this, we must account for the fact that we are counting both cakes, when we should just be counting one of them since they are identical!

Permutations with Repeats:
When we have n items and want to know how many ways we can order all n items given that there are r repeats of an item or several items, then:

$$\text{One Repeat: } \frac{n!}{r!} \text{ or Multiple Repeats: } \frac{n!}{r_1!r_2! \ldots r_n!}$$

Here r_1 represents the fact that item 1 is repeated r_1 times, r_2 represents the fact that item 2 is repeated r_2 times, and so on. Let's apply this formula to the above example.

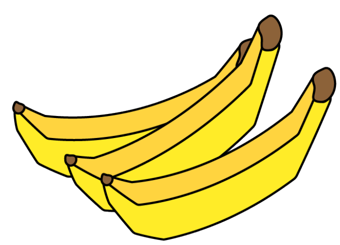
Solution: Here the numerator (4!) is simply from the numerator of our previous answer where there were no repeats. But since we have 2 repeats of the strawberry flavour:

$$\text{total number of different cakes} = \frac{4!}{2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1} = 12$$

By dividing by two, we essentially "got rid" of the extra identical cakes because we want the total number of **different** cakes.

Example: Using the letters in the word: **BANANA**

1. How many different 6-letter "words" can be formed?



Note: These do not have to be real words.

BANANA Example Video Solution:
https://youtu.be/aZli01zeyts

How Many Orders?

In problems about counting we often want to ask ourselves whether or not "order matters" in our choices. What is order in this case? And what does it mean? Let's take a look at two examples that can help us understand what "order" is.

Example 1: Ryan, Tim, Vince and Luc are in the finals for a race at the UW track meet. How many different ways can we award a first, second, and third place prize for the race? List all of the possible ways that the top 3 prizes can be awarded.

Solution: Let's write out all possible ways to award the top 3 prizes as "First Second Third" where R represents Ryan, T - Tim, V - Vince, L - Luc:

$R\ L\ V$	$R\ V\ L$	$R\ L\ T$	$R\ T\ L$	$R\ T\ V$	$R\ V\ T$
$L\ R\ V$	$L\ V\ R$	$L\ V\ T$	$L\ T\ V$	$L\ T\ R$	$L\ R\ T$
$V\ R\ T$	$V\ T\ R$	$V\ L\ T$	$V\ T\ L$	$V\ L\ R$	$V\ R\ L$
$T\ R\ V$	$T\ V\ R$	$T\ L\ R$	$T\ R\ L$	$T\ L\ V$	$T\ V\ L$

Thus, there 24 different ways to award a first, second, and third place prize.

Example 2: A national track team wants to make a team from Ryan, Tim, Vince and Luc at UW. How many different three-man teams can be made?

Solution: Let's write out all possible teams that can be made using the four athletes:

$R\ L\ V$	$R\ L\ T$	$R\ V\ T$	$V\ T\ L$

Thus, there are 4 different teams that could be made of only three athletes.

In both questions we looked at taking three students from four available students. But while they seem to follow the same idea, we ended up with different answers!

Let's take a closer look at the two examples and see how the order of their items affected the answer:

- **Example 1:**

  In the race there is a first, second, and third place prize. If the final outcome was R-T-V, this means that Ryan came first, Tim came second, and Vince came third. R-V-T would NOT be the same as R-T-V because now Vince came second, and Tim came in third. We can say that **ORDER MATTERS**.

- **Example 2:**

  In constructing a three-man team, if one of the athletes made the team, then they're on the team. There is no placing or hierarchy on the team. This means that the team R-V-T is the same as R-T-V, T-V-R, T-R-V, V-R-T and V-T-R. We can say that **ORDER DOESN'T MATTER**.

This is actually the difference between **Permutations** and **Combinations**!

> In Permutations **order matters**
>
> In Combinations **order doesn't matter**

Knowing when order does and doesn't matter is an important part of working with Permutations and Combinations so let's get some practice before we introduce Combinations.

Example: For each of the following scenarios, state whether or not order matters:

1. The number of ways three distinct plants can be arranged on a window sill.
2. Mr. Elgoog is asked to draw three cards from a deck of cards. In how many ways can he select three cards?
3. A math student is given a list of 8 problems and is asked to solve any 5 of the problems. How many different selections can the student make?
4. Selecting a 4-digit PIN code for a credit card.
5. The National Quidditch team is holding tryouts for one Keeper, one Seeker, and one Chaser. Seven students try out. How many ways can the three positions be filled?

Orders Example Video Solution: *https://youtu.be/NLYMW1E5l-g*

Note: In all of the Permutation problems we solved last week, order mattered!

Combinations

What if we had n objects in total and needed to choose k with **no repetition** (just like in a permutation) but now **order does not matter**?

Combinations are a way of counting in this type of situation when there is no repetition and **order doesn't matter**.

$$_nC_k = \frac{n!}{k!(n-k)!}$$

This is read as "n choose k" and counts how many ways we can choose k objects from a total of n objects.

We divide $n!$ by $k!$ because there are $k!$ ways to order the chosen k objects when order doesn't matter. If we do not divide by $k!$, we are counting each combination $k!$ times and we will not have the correct solution. We also divide $n!$ by $(n-k)!$ because again, we do not want to count the remaining $(n-k)$ objects that were not selected (we saw this in the Permutation formula as well).

Note: We can also write $_nC_k = \binom{n}{k}$

Example: Chocolate Frog Cards

Harry loses a bet to Ron and has to give Ron **4** of his chocolate frog **cards**. If Harry has **49 different cards** in his collection, how many ways can Ron pick which cards he'll take?

1. How many cards in total do we have to choose from and how many do we need to choose?

2. Does order matter when we choose our cards? Why?

3. Is repetition allowed in our choices? Why?

4. Use the combination formula to find how many ways Ron can pick the cards.

Solution:

1. There are 49 cards in total to choose from and we need to choose 4.
2. No, order does not matter because Ron ends up with the same 4 cards no matter which he picks first, second, third, and fourth.
3. No, there is no repetition allowed because once a card is picked, it can't be picked a second time.
4. Since **order doesn't matter** this is a Combinations problem. There are 49 total cards to choose from so $n = 49$. Since Ron needs to choose 4 cards, $k = 4$. Using the Combinations formula we get:

$$_{49}C_4 = \frac{49!}{4!(49-4)!} = \frac{49!}{4!45!} = \frac{49 \times 48 \times 47 \times 46}{4!} = 211,876$$

Thus there are 211, 876 ways Ron can pick the cards.

Example: Student Committee

There are 16 students on your school's student committee and there are 3 available executive positions. How many different ways can 3 students be elected from the student committee?

Solution:

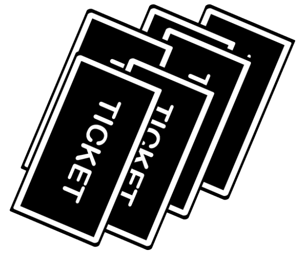
We are electing 3 students from the student committee of 16 students thus $n = 16$ and $k = 3$. The order in which these students are elected does not matter so we use the Combinations formula:

$$_{16}C_3 = \frac{16!}{3!(16-3)!} = \frac{16!}{3!13!} = \frac{16 \times 15 \times 14}{3!} = 560$$

There are **560** different ways to elect 3 students from the student committee.

Example: Movie Night

Suppose you just won 7 free movie tickets. You want to bring along 6 friends, but unfortunately, you have 10 friends who want to come along!



1. Does order matter?
2. Is repetition allowed?
3. How many different groups of friends could you take with you?
4. To make it a little fair, you decide you want to bring along exactly 3 girls and 3 boys. From the group of 10 friends who want to come along, there are 4 boys and 6 girls. How many ways can you pick 3 boys AND 3 girls?

Movie Night Example Video Solution: *https://youtu.be/jOQpA-zAx_U*

Pascal's Triangle

In the 16^{th} century, **Pascal's Triangle** was named after the French mathematician Blaise Pascal because of his work but interestingly enough, Pascal was definitely not the first to arrange these numbers into a triangle. It was worked on by Jia Xian in the 11^{th} century in China, and then it was popularized in the 13^{th} century by Chinese mathematician, Yang Hui and became known as *Yang Hui's Triangle.* Yet again, even before Yang Hui, it was discussed and known in the 11^{th} century as the *Khayyam Triangle* in Iran and was named after the Persian mathematician Omar Khayyam.

Isn't cool how people are able to study and discover mathematics from different parts of the world?

For this lesson, we will call it *Pascal's Triangle.* The triangle is built as follows:

row 0 ⟶ 1
row 1 ⟶ 1 1
row 2 ⟶ 1 2 1
row 3 ⟶ 1 3 3 1
row 4 ⟶ 1 4 6 4 1
row 5 → 1 5 10 10 5 1
↑ ↑ ↑ ↑ ↑ ↑
entry 0 entry 1 entry 2 entry 3 entry 4 entry 5

Note: In Pascal's Triangle the rows start at index 0. Similarily, the entries in each row start at index 0.

Each number in Pascal's Triangle can be written as a combination as seen below.

$$\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}$$

In general, the k-th entry in the n-th row can be written as $\binom{n}{k} = {}_nC_k$.

That is, for the first few entries we have:

$$\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{array}
\longrightarrow
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1
\end{array}$$

Example: Find the number for each entry in Pascal's Triangle:

1. The 6th entry of the 8th row:

The 6th entry of the 8th row can be written as $\binom{8}{6} = {}_8C_6$. Thus:

$$\begin{aligned}\binom{8}{6} &= \frac{8!}{6!(8-6)!} \\ &= \frac{8\times7\times6\times5\times4\times3\times2\times1}{6\times5\times4\times3\times2\times1\times(2\times1)} \\ &= \frac{8\times7}{2} \\ &= 28\end{aligned}$$

Thus the 6th entry of the 8th row is 28.

2. The 0th entry of the 3rd row:

   The 0th entry of the 3rd row can be written as $\binom{3}{0} = {}_3C_0$. Thus:

$$\begin{aligned}\binom{3}{0} &= \frac{3!}{0!(3-0)!} \\ &= \frac{3\times2\times1}{3\times2\times1} \\ &= 1\end{aligned}$$

   Thus the 0th entry of the 3rd row is 1.

3. The 12th entry of the 10th row:

   We cannot find this because the 10th row only has entries from 0 to 10. It does not have a 12th entry!