



Grade 7/8 Math Circles
November 18, 2020
Introduction to Vectors - Solutions

Vectors and Scalars

What do you think about when you hear the word “size”? Most likely it’s about how large or how small an object is. The word size seems to imply how big something is. So which would you say is “bigger”: 4 km or 18 km? 15 seconds or 15 minutes? The speed of a bike or the speed of a train? All of these things have a property that we call “size” or “magnitude”. That is, they are quantifiable: we can assign a number to them and compare that quantity to other quantifiable things. 15 minutes is longer or “bigger” than 15 seconds, in a sense.

Let’s try something different: which quantity is “bigger”; 5 m to the left or 5 m to the right? Trick question! They both have the same size, but they *are* different – they have different *directions*.

A **vector** is a quantity with both a size and a direction, whereas a **scalar** is a quantity with only a size. For example, 5 m would be a scalar quantity because it has a size associated with it – 5 m is 5 m long. On the other hand, 5 m to the right is a vector quantity because there is both a size and a direction associated with it.

Example: Describe the following quantities using vector or scalar values as appropriate.

1. I need to travel to a bookstore that is East of my house and about a kilometer away.
2. Cindy is driving North, at 50 km/h.
3. Frank waited for an hour before his friend showed up.
4. Alice is hiking through a forest. She’s going directly South-East.
5. The ball fell at 10 m/s.

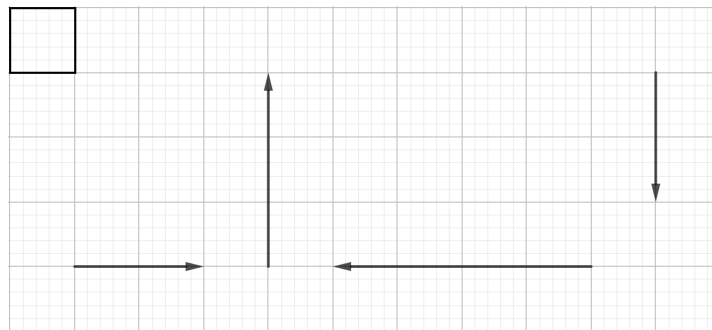
Video Solution: <https://youtu.be/89REZZSX3zY>

Drawing Vectors

Arrows also have a size and direction like vectors do. This is what we'll use to draw vectors:

- **Size** is represented by how long the arrow is.
- **Direction** is represented by which way the arrow is pointing.

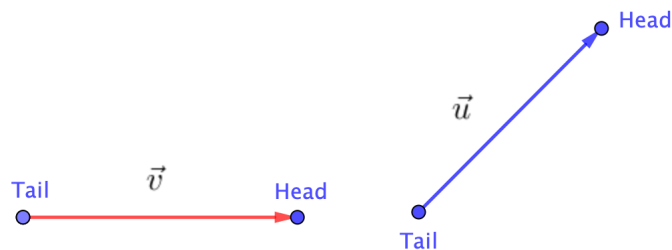
Example: Give the size and direction of each vector below. In this case, each square represents 1 unit of measurement (e.g. the bolded square represents 1 unit of measurement). Represent the directions as North, East, South, or West.



Video Solution: <https://youtu.be/NcmbbCzkIGQ>

Vector Operations: Addition

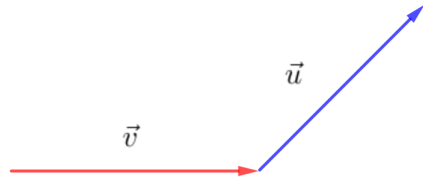
We can call the initial point of an arrow the **tail** of a vector, and the arrow head as the **head** of a vector.



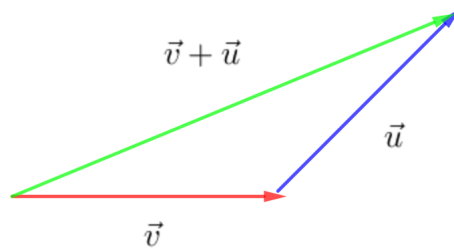
We can represent the red vector as \vec{v} , and the blue vector as \vec{u} . We draw a little arrow on top of the letters to show that they represent vectors.

Adding vectors together is as simple as following the arrows. If I want to add two vectors together, then I need to follow the first vector and then follow the second vector. This means

we start at the tail of the first vector, follow the vector to its head, then follow from the tail of the of the second vector, and then follow the vector to its head. Essentially, we attach the head of the first vector to the tail of the second vector. Adding the above vectors gives:



The result looks like this:



That is, the sum of two vectors is a vector. In this case, the green vector is equal to $\vec{v} + \vec{u}$. From the starting point (the tail of \vec{v}), the total “shift” to the end point (the head of \vec{u}), is represented in green. This vector also has a direction and size. For example, $\vec{v} + \vec{u}$ could have direction North-East and size equal to 6 cm.

Note: We loosely use the word “shift” since vectors can apply to a variety of things like velocity, displacement, acceleration, etc.

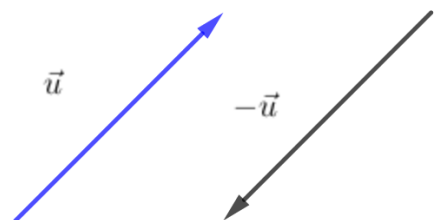
Remark: Equal vectors are vectors that have the same **size** and the same **direction**.

Example: Is $\vec{v} + \vec{u}$ equal to $\vec{u} + \vec{v}$?

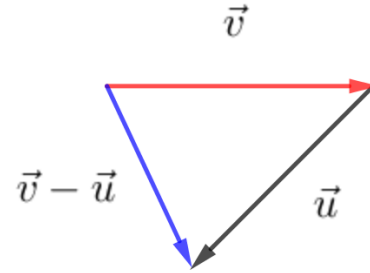
Video Solution: <https://youtu.be/ZcMz4X1S-J8>

Vector Operations: Subtraction

To subtract vectors we use a similar strategy. We follow vector \vec{v} , then go backwards along vector \vec{u} , by turning it around. We switch the direction of the vector being subtracted. If we have vector \vec{u} , then switching its direction gives $-\vec{u}$.

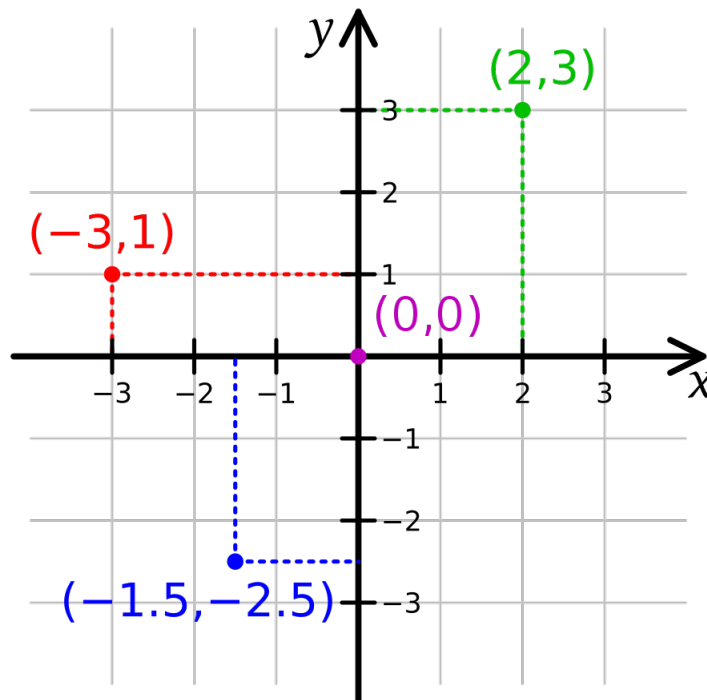


Thus to solve $\vec{v} - \vec{u}$, we switch the direction of \vec{u} and then attach the head of \vec{v} to the tail of \vec{u} . This gives:



2D Cartesian Coordinate System

Let's review a familiar coordinate system: the 2D Cartesian coordinate system.



You can see that the 2D cartesian coordinate system uses 2 variables to describe any location in the 2D space. This coordinate system is defined by two axes, a horizontal axis labeled x , and a vertical axis labeled y . Coordinate points are represented as (x, y) where x is the x -coordinate and y is the y -coordinate. We have the following coordinate points above: $(2, 3)$, $(-3, 1)$, $(0, 0)$, $(-1.5, -2.5)$.

Vectors and the 2D Coordinate System

We know that in a 2D coordinate system the coordinate (x, y) tells us two things: how far to go and where. For example, the coordinate $(2, 3)$ tell us to go 2 units along the x -axis, and 3 units along the y -axis. This is perfect for visualizing vectors! Starting from the tail of a vector, we want to know how far away the head of a vector is in the horizontal direction, and how far away it is in the vertical direction. A coordinate gives us **size** and **direction** so we can easily draw and describe vectors.

In a 2D Coordinate system we can represent the vector \vec{v} , as

$$\vec{v} = (x, y)$$

The (x, y) coordinate point represents the **direction** of a vector. Starting at the **tail** of a vector, the horizontal distance to the **head** of the vector is the x -value, and the vertical distance to the **head** of the vector is the y -value.

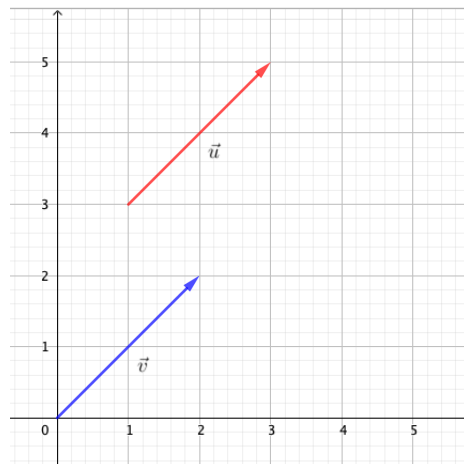
A vector drawn in **standard** form has its **tail** located at coordinate point $(0, 0)$. This point is called the **origin**; it's the starting point from which all other coordinate points are referenced from.

Example: Draw the following vectors in standard form: $\vec{u} = (2, 5)$, $\vec{v} = (1, 1)$, $\vec{w} = (-4, 2)$, and $\vec{z} = (-2, -4)$.

Video Solution: https://youtu.be/5fH_jyacFkE

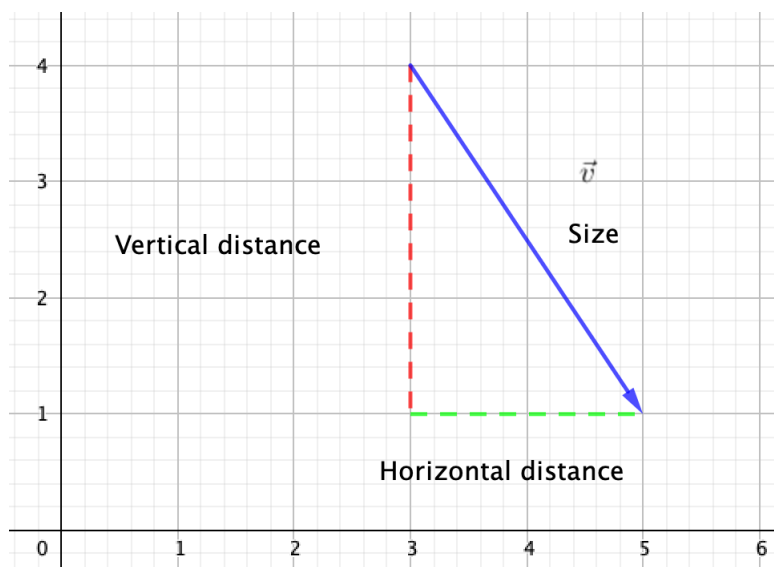
Example: Consider the following two vectors. Are they equivalent?

Solution: Yes! – Can you think of why? Remember, vectors have **size** and **direction**. In this case, both vectors are pointing in the exact same direction and they have equal size (we can see this visually as the length of the arrows). Here \vec{v} is simply drawn in standard form but it is still equivalent to \vec{u} , where $\vec{v} = (2, 2) = \vec{u}$.



Vector Size from a Graph

As we mentioned earlier, a vector is defined by both its **size** and its **direction**. Using a graph, how can we calculate the size of a vector? Remember that the size of a vector is represented by the length of the arrow. In other words, it's the distance between the tail and the head of the vector. Consider the following vector, \vec{v} :



Here the **size** of \vec{v} is the length of the arrow in blue. We can represent the **vertical distance** from the tail of \vec{v} to the head of \vec{v} in red, and the **horizontal distance** from the tail of \vec{v} to the head of \vec{v} in green.

Question: How many units is the horizontal and vertical distance of \vec{v} ?

Answer: In this case, the horizontal distance is 2 units and the vertical distance is 3 units. Can you see why?

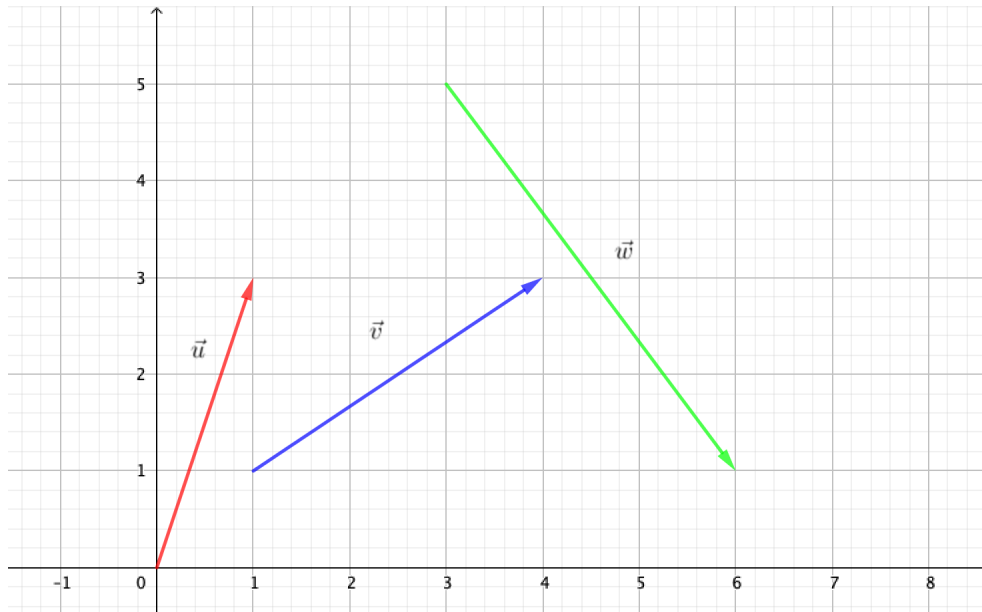
We can calculate the **size** of a vector $\vec{v} = (x, y)$, using the following equation:

$$\|\vec{v}\| = \sqrt{(x)^2 + (y)^2}$$

In this case, x represents the **horizontal distance** of the vector \vec{v} and y represents the **vertical distance** of the vector \vec{v} .

We use two vertical lines on both sides of a vector to represent its **size**.

Example: Calculate the size of the following vectors \vec{u} , \vec{v} , and \vec{w} .



Video Solution: <https://youtu.be/9d7RjxwBy5c>

Note: This formula comes from a special relationship known as the **Pythagorean Theorem**.

Use the following GeoGebra activity to visualize and create your own vectors: <https://www.geogebra.org/m/f6faacjg>

Problem Set

1. What is the difference between a scalar and a vector? What two things will a vector tell you? Give an example of each.

Answers vary.

2. For each of the following, state whether the quantity given is a scalar or a vector.

- (a) That car was going 40 km/h over the speed limit!

Scalar: 40km/h

- (b) Jason moved his desk 5 m.

Scalar: 5m

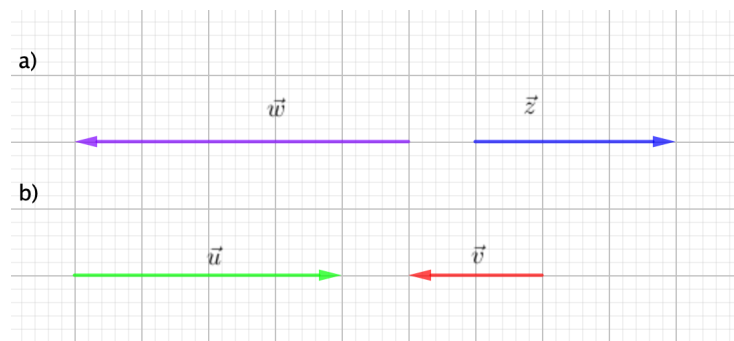
- (c) Toronto is about 120 km East from Waterloo.

Vector: Toronto is 120 km [East] of Waterloo (or just 120 km if you didn't have a map)

- (d) The line is 7 cm long.

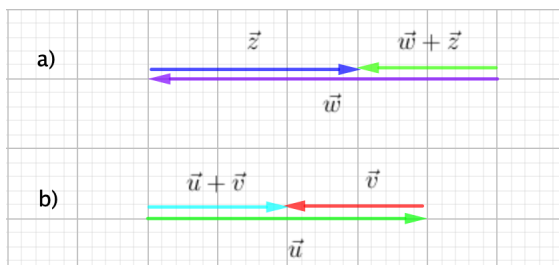
Scalar: 7m

3. Find $\vec{w} + \vec{z}$ and $\vec{u} + \vec{v}$. Use the GeoGebra activity linked at the end of the lesson to create each vector and find the sum. State the length of each sum, in this case each square is one unit in length. State the direction of each sum in terms of North, East, South, or West.



a) Size = 2. Direction is West.

b) Size = 2. Direction is East.

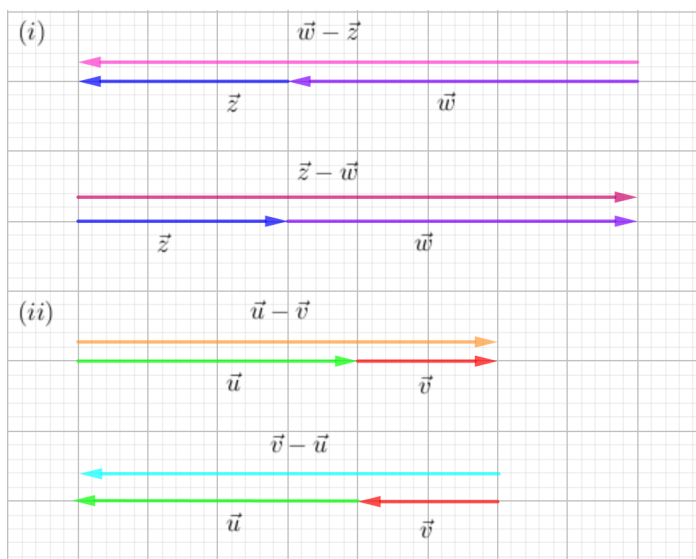


4. Using the same vectors in 3a) and 3b) above:

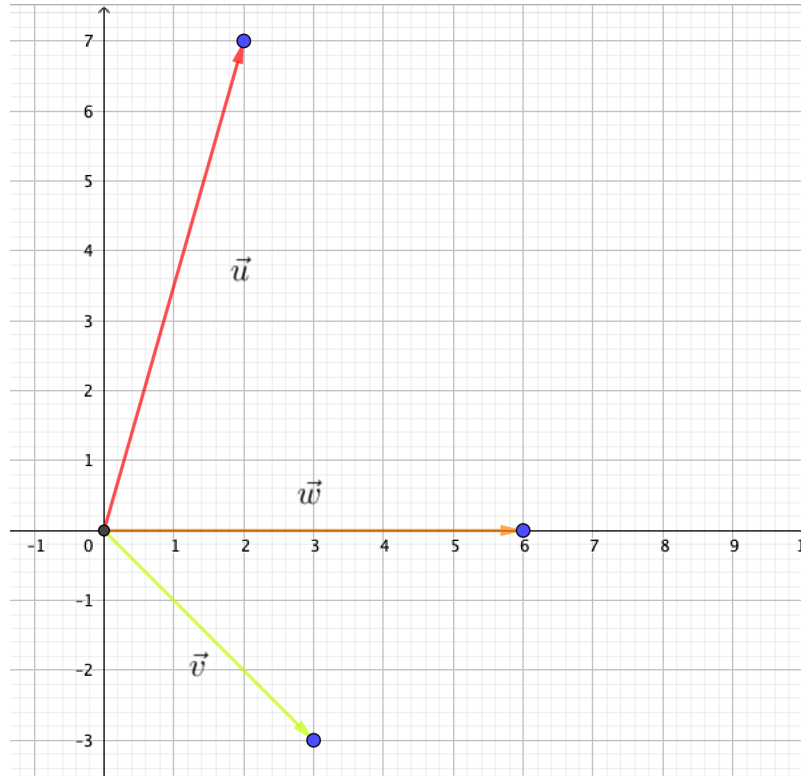
- (i) Find $\vec{w} - \vec{z}$ and $\vec{z} - \vec{w}$. State the length of the each difference, each square is one unit in length. State the direction of each difference in terms of North, East, South, or West.
- (ii) Find $\vec{u} - \vec{v}$ and $\vec{v} - \vec{u}$. State the length of the each difference, each square is one unit in length. State the direction of each difference in terms of North, East, South, or West.

(i) $\vec{w} - \vec{z}$: Size = 8. Direction is West. $\vec{z} - \vec{w}$: Size = 8. Direction is East.

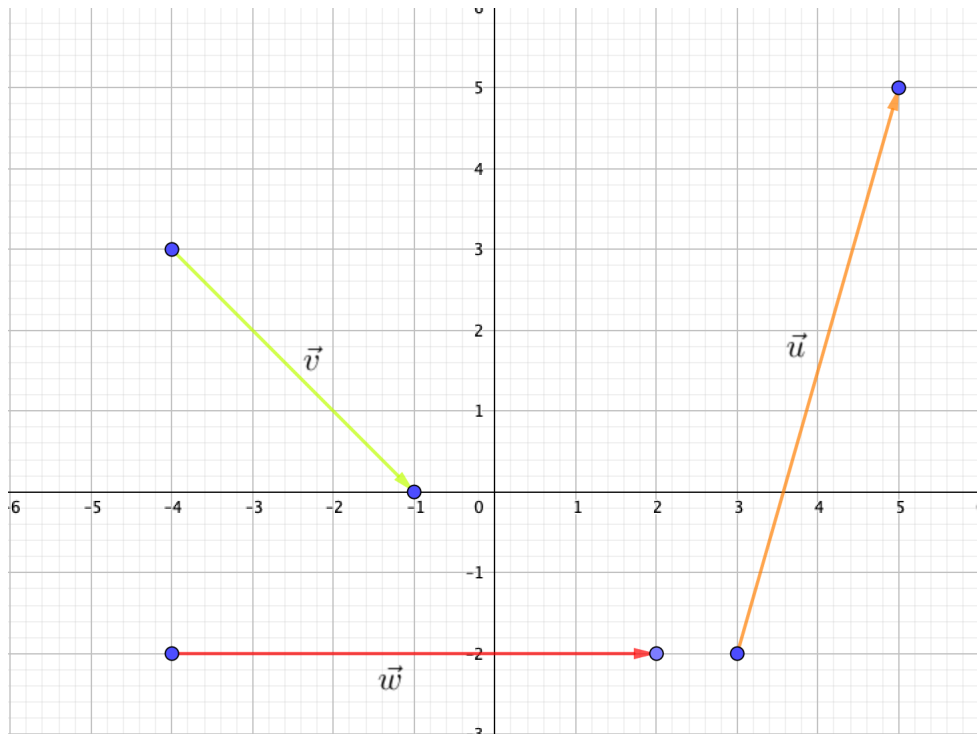
(ii) $\vec{u} - \vec{v}$: Size = 6. Direction is East. $\vec{v} - \vec{u}$: Size = 6. Direction is West.



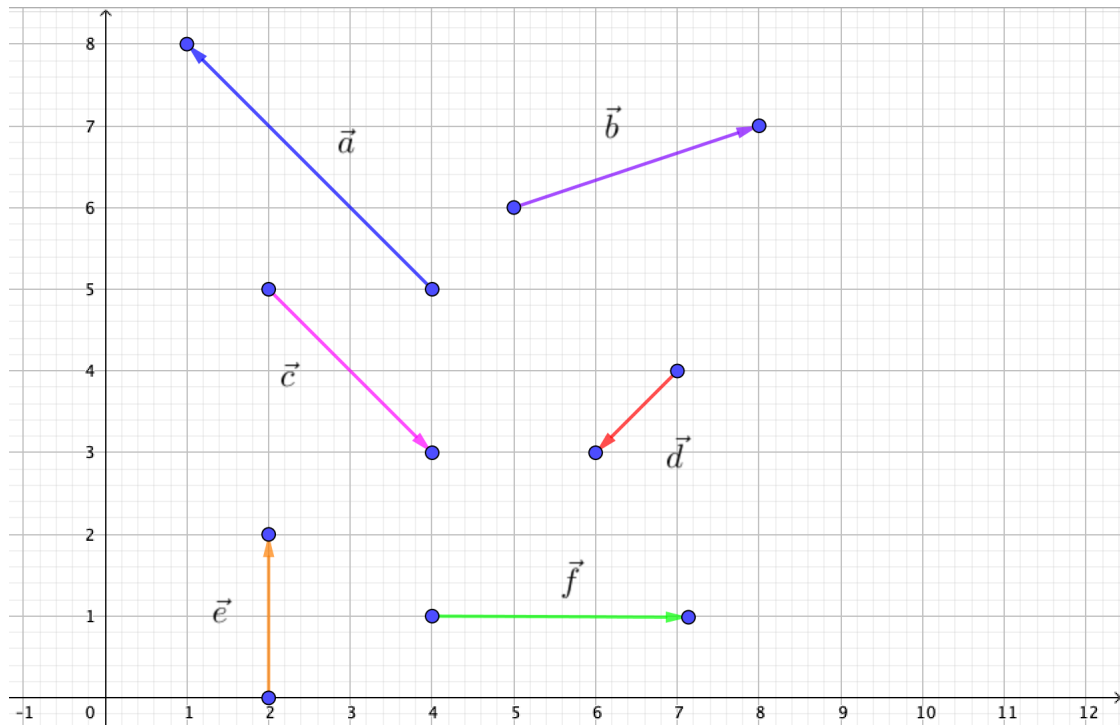
5. Use the GeoGebra activity to draw the following vectors in standard form: $\vec{u} = (2, 7)$, $\vec{v} = (3, -3)$, and $\vec{w} = (6, 0)$.



6. Use the GeoGebra activity to draw the following vectors **not** in standard form: $\vec{u} = (2, 7)$, $\vec{v} = (3, -3)$, and $\vec{w} = (6, 0)$.



7. Use the GeoGebra activity to create the following vectors to solve the following:



State each result as a vector. For example if $\vec{a} + \vec{b} = \vec{r}$, then $\vec{r} = (x, y)$.

Let the vector \vec{s} be the solution to each result.

(i) $\vec{a} - \vec{b}$

$$\vec{s} = (-6, 2)$$

(ii) $\vec{d} + \vec{c}$

$$\vec{s} = (1, -3)$$

(iii) $\vec{d} - \vec{f}$

$$\vec{s} = (-1, -4)$$

(iv) $\vec{e} + \vec{f}$

$$\vec{s} = (3, 2)$$

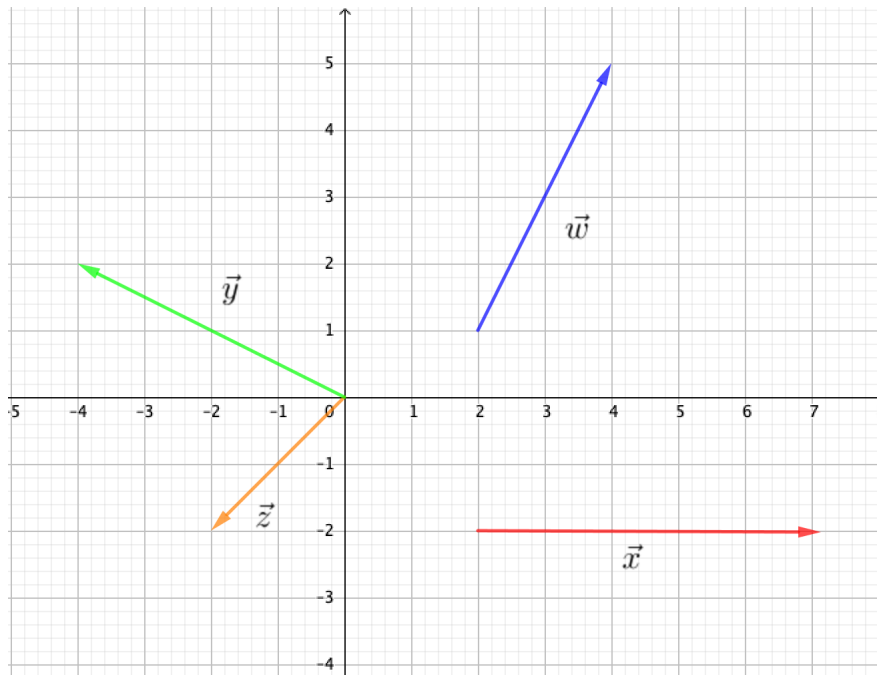
8. Calculate the size of each of the following vectors: $\vec{u} = (2, 7)$, $\vec{v} = (3, -3)$, and $\vec{w} = (6, 0)$. Use a calculator to show the size up to 2 decimal places.

$$\|\vec{u}\| = \sqrt{(2)^2 + (7)^2} \approx 7.28$$

$$\|\vec{v}\| = \sqrt{(3)^2 + (-3)^2} \approx 4.24$$

$$\|\vec{w}\| = \sqrt{(6)^2 + (0)^2} = 6.00$$

9. Calculate the size of each of the following vectors. Use a calculator to show the size up to 2 decimal places.



$$\|\vec{w}\| = \sqrt{(2)^2 + (4)^2} \approx 4.47$$

$$\|\vec{x}\| = \sqrt{(5)^2 + (0)^2} = 5.00$$

$$\|\vec{y}\| = \sqrt{(-4)^2 + (2)^2} \approx 4.47$$

$$\|\vec{z}\| = \sqrt{(-2)^2 + (-2)^2} \approx 2.83$$