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Grade 7/8 Math Circles

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Introduction to Interest - Solution

Introduction

Interest can be seen as a reward for lending your money to a bank or other business. It can also be seen as a price for borrowing money. Suppose I take a loan from a bank. When I go to pay this money back, I'll have to pay the amount I orginally borrowed, plus a fee that is caclulated as the cost of borrowing the money. For example, if I borrowed \$1000 with a \$100 cost to borrow, I will have to pay the bank back a total of \$1100. This \$100 is called interest, and is usually calculated as a rate.

Interest

Interest rates are expressed as percentages and must be converted into decimals in order to be used in equations. To convert a percentage into a decimal, divide the percentage by 100. This can be simply done using a calculator, or by moving the decimal two places to the left.

$$75\% \longrightarrow 75.0 \longrightarrow 0.75$$

Example: Convert the following percentages into decimals: 53%, 8%. 27.6%, and 0.6%.

Solution:

1. 53%	$53 \div 100 = 0.53$	$53\% \longrightarrow 53.0 \longrightarrow 0.53$
2. 8%	$8 \div 100 = 0.08$	$8\% \longrightarrow 08.0 \longrightarrow 0.08$
3. 27.6%	$27.6 \div 100 = 0.276$	$27.6\% \longrightarrow 27.6 \longrightarrow 0.276$
4. 0.6%	$0.6 \div 100 = 0.006$	$0.6\% \longrightarrow 00.6 \longrightarrow 0.006$

To calculate a percentage of a number, multiply the number by the decimal form of the percentage.

Example: Calculate 25% of 700.

Solution:

$$0.25 \times 700 = 175$$

So 25% of 700 is 175.

Let's take a look at some examples on calculating interest using different rates over specific periods of time.

Simple Interest

Under **simple interest**, the interest earned (or paid) is calculated by multiplying the original amount invested (or borrowed), called the **principal**, by the annual (yearly) interest rate. This is then multiplied by the length of the investment (or loan) in years.

> The formula to calculate simple interest, I, is:
>
> $$I = Prt$$
>
> where P is the **p**rincipal of the loan (or investment), r is the interest **r**ate, and t is the **t**ime in years.

Note: Prt is equivalent to $P \times r \times t$.

Example: What is the simple interest due on a \$2500 loan at the end of 10 months if the annual interest rate is 7.5%

Solution: To begin, it is a good idea to list the information that we need and the information that we are given. In this case, we want to calculate simple interest, I, and we are given the principal P, rate r, and time t. In particular,

$$\begin{aligned} I &= ? \\ P &= 2500 \\ r &= 7.5\% \\ t &= 10 \text{ months} \end{aligned}$$

First of all, we need to convert our rate into a decimal.

$$7.5\% = 0.075$$

Recall simple interest is based on time in **years**. This means we cannot simply plug 10 into our equation since it is months. We need to convert months into years. In this case,

$$\text{10 months is equivalent to } \frac{10}{12} \text{ of a year}$$

Now we can subsitiute the information we have into our simple interest equation.

$$\begin{aligned} I &= Prt \\ &= (2500)(0.075)\left(\frac{10}{12}\right) \\ &= 156.25 \end{aligned}$$

Thus the total interest due at the end of 10 months is \$156.25.

Accumulated value, denoted by S, is the **total** amount of a loan or investment after t years. Under simple interest, the accumulated value is the sum of the principle amount and the interest amount.

$$S = P + I$$

Since $I = Prt$, this can be written as:

$$S = P + Prt = P(1 + rt)$$

Note: $P(1 + rt)$ is equivalent to $P \times (1 + rt)$

Aside: How is $P + Prt = P(1 + rt)$? Let's take a further look: *https://youtu.be/RiThQP7_MU0*

> Under simple interest, the formula to calculate the accumulated value, S, is:
>
> $$S = P(1 + rt)$$
>
> where P is the **p**rincipal of the loan (or investment), r is the interest **r**ate, and t is the **t**ime in years.

Example: Suppose I took a \$5000 loan from a bank which earns 6% simple interest annually. How much money will I owe if I pay back the loan in ten years? How much is the total interest I owe in 10 years?

Video Solution: *https://youtu.be/78iV0aCFGrQ*

What if you are asked to solve a problem where you are given the **accumulated value**, but not the **principal amount**? This is called a **present value calculation**. To solve this we can rearrange our simple interest equation to solve for P instead of S. As a matter of fact, you can use your equation solving skills to solve for any unknown.

Example: How much would I have to deposit into my savings account today in order to have \$1600 in 3 years if the account earns 5% simple interest annually?

Video Solution: *https://youtu.be/3334njv2wQk*

Before we look at another type interest, let's take look at a method of calculation that we will need for our next formula.

Aside: Exponentiation

Exponentiation is *repeated multiplication.*

An exponent and base looks like the following:

$$4^3$$

The small number written above and to the right of the number is called the **exponent**. The number underneath the exponent is called the **base**. In the example, 4^3, the exponent is 3 and the base is 4. We say "4 is raised to the **third power**".

An exponent tells us to multiply the base by itself that number of times. In the example above, $4^3 = 4 \times 4 \times 4$. Once we write out the multiplication problem, we can easily evaluate the expression. Let's do this for the example we've been working with:

$$4^3 = 4 \times 4 \times 4 = 16 \times 4 = 64$$

The main reason we use exponents is because it's a shorter way to write out big numbers. For example, we can express the following long expression: $2 \times 2 \times 2 \times 2 \times 2 \times 2$ as 2^6 since 2 is being multiplied by itself 6 times. We say 2 is raised to the **6th power**.

Exercise: Evaluate.

$2^5 =$ ________ $1^9 =$ ________ $(1+3)^3 =$ ________ $(4+4)^2 =$ ________

Video Solution: *https://youtu.be/k8Q0Tojmz_I*

Compound Interest

Under **compound interest**, the interest earned (or paid) is calculated on the original amount invested (or borrowed) **and** the interest already earned (or paid), during the length of the loan (or investement) in years.

Example: Assume that you have deposited \$1000 in a savings account that pays 10% interest compounded annually.

- After one year, the money earns $1000 \times 0.1 = \$100$.

  The new balance is $\$1100 = 1000 \times (1 + 0.1)$.

- After two years, the money earns $1100 \times 0.1 = \$110$.

  The new balance is $\$1210 = 1100 \times (1 + 0.1) = 1000 \times (1 + 0.1) \times (1 + 0.1) = 1000 \times (1 + 0.1)^2$.

- After three years, the money earns $1210 \times 0.1 = \$121$.

  The new balance is $\$1331 = 1210 \times (1+0.1) = 1000 \times (1+0.1) \times (1+0.1) \times (1+0.1) = 1000 \times (1 + 0.1)^3$.

And so on.

> Under compound interest, the formula to calculate accumulated value, S, is:
>
> $$S = P(1 + r)^t$$
>
> where P is the **p**rincipal of the loan, r is the interest **r**ate, and t is the **t**ime in years.

Example: Let's go back to the example above. We saw that under compound interest, the balance in the savings account after 3 years would be \$1331. What would the balance be in the account after 3 years if it was under 10% **simple interest**?

Solution: Recall that the forumla to calculate accumulated value under simple interest is $S = P(1 + rt)$.

We have that the principal amount P, is \$1000. The rate of interest r, is $10\% = 0.10$, and time t, is 3 years. Then,

$$S = P(1 + rt) = 1000(1 + (0.10)(3)) = 1300$$

The balance in the savings account after 3 years under simple interest is \$1300.

Which interest type would you rather want for your savings account, compound or simple? I would want compound interest! After 3 years under simple interest, I earned \$300 in interest (\$1000 + 300). Whereas after 3 years under compound interest, I earned \$331 of interest (\$1000 + 331). In this case, I earn more money if my interest is compounded annually!

Examples:

1. I deposit \$5000 into a savings account which earns 6% interest compounded annually. How much money will be in the account in ten years?

2. How much do I have to deposit today in order to have \$5000 in 5 years if interest is 8.5% compounded annually?

Video Solution: *https://youtu.be/O2Lq6svbQjE*

Nominal Rates

So far we have only used compound interest rates that are compounded annually. The term "compounded annually" means that the interest is calculated and applied once a year. More frequent periods are also available. For example, interest could be calculated twice a year, four times a year, every month, every week, every day, etc. Because of this, we need to make some small changes to our formula to calculate the accumulated value under compound interest:

Under compound interest, the formula to calculate accumulated value, S, is:

$$S = P\left(1 + \frac{r}{m}\right)^n$$

where m is the number of compounding periods in a year, r is the nominal rate of interest compounded m times a year, and n is the number of compounding periods over the entire life of the loan.

In the context of a question, how do you know what m is?

Ask yourself: "How many compounding periods can fit in one year?"

- "Compounded annually" $\Rightarrow m = 1$

- "Compounded semi-annually" $\Rightarrow m = 2$
- "Compounded quarterly" $\Rightarrow m = 4$
- "Compounded monthly" $\Rightarrow m = 12$
- "Compounded weekly" $\Rightarrow m = 52$
- "Compounded daily" $\Rightarrow m = 365$

Note: If t represents the number of years (as we've seen before), then we know that since n is the number of compounding periods over the entire life of the loan, we have $\mathbf{n} = \mathbf{m} \times \mathbf{t}$.

For example, if t is equal to 5 years and interest is compounded monthly, then the total number of compounding periods over 5 years, would be $n = 12 \times 5 = 60$.

Example: I deposit \$5000 into a savings account which earns 6% interest compounded monthly. How much money will be in the account in ten years?

Video Solution: *https://youtu.be/l931nkDd3J8*

Problem Set - Solutions

1. Determine the accumulated values of the following loans.

   (a) A \$1200 loan for 7 months at 5% simple interest.

   $S=?\qquad P=1200\qquad r=0.05\qquad t=\dfrac{7}{12}$

   $$\begin{aligned} S &= P(1+rt)\\ &= 1200\left(1+(0.05)\left(\frac{7}{12}\right)\right)\\ &= 1235 \end{aligned}$$

   The accumulated value is \$1235.

   (b) An \$8000 loan for 4 years at 12.5% simple interest.

   $S=?\qquad P=8000\qquad r=0.125\qquad t=4$

   $$\begin{aligned} S &= P(1+rt)\\ &= 8000(1+(0.125)(4))\\ &= 12000 \end{aligned}$$

   The accumulated value is \$12000

   (c) A \$500 loan for 99 days at 10% simple interest. (Note: there are 365 days in a year.)

   $S=?\qquad P=500\qquad r=0.1\qquad t=\dfrac{99}{365}$

   $$\begin{aligned} S &= P(1+rt)\\ &= 500\left(1+(0.1)\left(\frac{99}{365}\right)\right)\\ &= 513.56 \end{aligned}$$

   The accumulated value is \$513.56.

   (d) A \$750 loan for 15 weeks at 13.25% simple interest. (Note: there are 52 weeks in a year.)

$S = ? \qquad P = 750 \qquad r = 0.1325 \qquad t = \frac{15}{52}$

$$\begin{aligned} S &= P(1 + rt) \\ &= 750\left(1 + (0.1325)\left(\frac{15}{52}\right)\right) \\ &= 778.67 \end{aligned}$$

The accumulated value is \$778.67.

2. Determine the principal value if a savings account holds \$3600 after 10 years at 8% simple interest.

   $S = 3600 \qquad P = ? \qquad r = 0.08 \qquad t = 10$

$$\begin{aligned} S &= P(1 + rt) \\ P &= \frac{S}{1 + rt} \\ &= \frac{3600}{1 + (0.08)(10)} \\ &= 2000 \end{aligned}$$

   The principal/present value is \$2000.

3. A loan of \$100 is to be repaid with \$120 at the end of 10 months. What is the annual simple interest rate?

   $P = 100 \qquad I = 20 \qquad r = ? \qquad t = \frac{10}{12}$

$$\begin{aligned} I &= Prt \\ \frac{I}{Pt} &= \frac{Prt}{Pt} \\ r &= \frac{I}{Pt} \\ &= \frac{20}{(100)\left(\frac{10}{12}\right)} \\ &= 0.24 \end{aligned}$$

   The annual simple interest rate is 24%.

   Using the other equation:

$P = 100 \qquad S = 120 \qquad r = ? \qquad t = \frac{10}{12}$

$$
\begin{aligned}
S &= P(1 + rt) \\
\frac{S}{P} &= \frac{P(1 + rt)}{P} \\
\frac{S}{P} - 1 &= 1 + rt - 1 \\
\frac{\frac{S}{P} - 1}{t} &= \frac{rt}{t} \\
r &= \frac{\frac{S}{P} - 1}{t} \\
&= \frac{\frac{120}{100} - 1}{\frac{10}{12}} \\
&= 0.24
\end{aligned}
$$

The annual simple interest rate is 24%.

4. How long will it take \$3000 to earn \$60 interest at 6% simple interest?

   $P = 3000 \qquad I = 60 \qquad r = 0.06 \qquad t = ?$

$$
\begin{aligned}
I &= Prt \\
\frac{I}{Pr} &= \frac{Prt}{Pr} \\
t &= \frac{I}{Pr} \\
&= \frac{60}{(3000)(0.06)} \\
&= 0.333... \\
&= \frac{1}{3}
\end{aligned}
$$

   It will take $\frac{1}{3}$ of a year (4 months) to earn \$60 interest.

5. Determine the accumulated values of the following loans.

   (a) A \$2000 loan for 4 years at 5% interest compounded annually.

$S = ? \qquad P = 2000 \qquad r = 0.05 \qquad t = 4$

$$\begin{aligned} S &= P(1+r)^t \\ &= 2000(1+0.05)^4 \\ &= 2431.01 \end{aligned}$$

The accumulated value is \$2431.01.

(b) A \$100 loan for 25 years at 7.5% interest compounded annually.

$S = ? \qquad P = 100 \qquad r = 0.075 \qquad t = 25$

$$\begin{aligned} S &= P(1+r)^t \\ &= 100(1+0.075)^{25} \\ &= 609.83 \end{aligned}$$

The accumulated value is \$609.83.

6. Determine the principal value:

   (a) If a savings account holds \$7500 after 10 years at 8% interest compounded annually.

   $S = 7500 \qquad P = ? \qquad r = 0.08 \qquad t = 10$

$$\begin{aligned} S &= P(1+r)^t \\ P &= \frac{S}{(1+r)^t} \\ &= \frac{7500}{(1+0.08)^{10}} \\ &= 3473.95 \end{aligned}$$

   The principal/present value is \$3473.95.

   (b) If a savings account holds \$25000 after 50 years at 4.5% interest compounded annually.

$S = 25000 \qquad P = ? \qquad r = 0.045 \qquad t = 50$

$$\begin{aligned} S &= P(1+r)^t \\ P &= \frac{S}{(1+r)^t} \\ &= \frac{25000}{(1+0.045)^{50}} \\ &= 2767.74 \end{aligned}$$

The principal/present value is \$2767.74.

7. Determine what amount must be invested at a rate of 5% to accumulate $S = \$5000$ at the end of four years under

   (a) simple interest;

   (b) compound interest (compounded annually).

$S = 5000 \qquad P = ? \qquad r = 0.05 \qquad t = 4$

(a)

$$\begin{aligned} S &= P(1+rt) \\ P &= \frac{S}{(1+rt)} \\ &= \frac{5000}{(1+(0.05)(4))} \\ &= 4166.67 \end{aligned}$$

You must invest \$4166.67.

(b)

$$\begin{aligned} S &= P(1+r)^t \\ P &= \frac{S}{(1+r)^t} \\ &= \frac{5000}{(1+0.05)^4} \\ &= 4113.51 \end{aligned}$$

You must invest \$4113.51.

8. Determine the accumulated values of the following loans.

   (a) A \$1000 loan for 3 years at 13% interest compounded weekly.

   $S = ?$ $\qquad P = 1000$ $\qquad m = 52$ $\qquad r = 0.13$ $\qquad n = 52 \times 3 = 156$

$$\begin{aligned} S &= P\left(1+\frac{r}{m}\right)^n \\ &= 1000\left(1+\frac{0.13}{52}\right)^{156} \\ &= 1476.26 \end{aligned}$$

   The accumulated value is \$1476.26.

   (b) A \$500 loan for 25 years at 4% interest compounded semi-annually.

   $S = ?$ $\qquad P = 500$ $\qquad m = 2$ $\qquad r = 0.04$ $\qquad n = 2 \times 25 = 50$

$$\begin{aligned} S &= P\left(1+\frac{r}{m}\right)^n \\ &= 500\left(1+\frac{0.04}{2}\right)^{50} \\ &= 1345.79 \end{aligned}$$

   The accumulated value is \$1345.79.

9. Determine the principal value:

   (a) If a savings account holds \$6000 after 10 years at 15% interest compounded quarterly.

   $S = 6000$ $\qquad P = ?$ $\qquad m = 4$ $\qquad r = 0.15$ $\qquad n = 4 \times 10 = 40$

$$\begin{aligned} S &= P\left(1+\frac{r}{m}\right)^n \\ P &= \frac{S}{\left(1+\frac{r}{m}\right)^n} \\ &= \frac{6000}{\left(1+\frac{0.15}{4}\right)^{40}} \\ &= 1376.03 \end{aligned}$$

   The principal/present value is \$1376.03.

(b) If a savings account holds \$25000 after 50 years at 12% interest compounded monthly.

$S = 25000 \qquad P = ? \qquad m = 12 \qquad r = 0.12 \qquad n = 12 \times 50 = 600$

$$
\begin{aligned}
S &= P\left(1 + \frac{r}{m}\right)^n \\
P &= \frac{S}{\left(1 + \frac{r}{m}\right)^n} \\
&= \frac{25000}{\left(1 + \frac{0.12}{12}\right)^{600}} \\
&= 63.84
\end{aligned}
$$

The principal/present value is \$63.84.