Intermediate Math Circles

October 14, 2020

Perfect Squares

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www.cemc.uwaterloo.ca

Perfect Squares

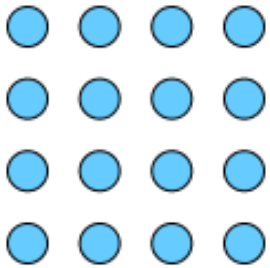
A perfect square is an integer that is the square of an integer.

In other words, an integer s is a perfect square if $s = n^2$ for some integer n.

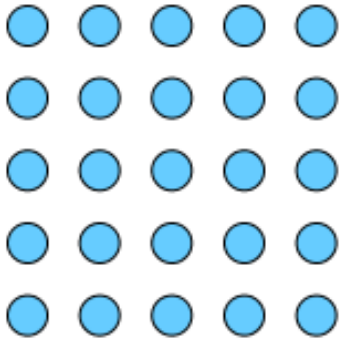
There are many different ways to illustrate a perfect square and they often involve the geometric notion of a square.

Perfect Squares

16 is a perfect square since $16 = 4^2$. We can illustrate this perfect square by drawing 16 dots arranged in a 4×4 square grid.

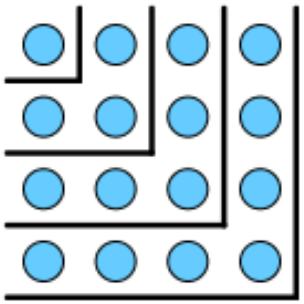


Similarly, we can illustrate the perfect square $25 = 5^2$ by drawing 25 dots arranged in a 5×5 square grid.

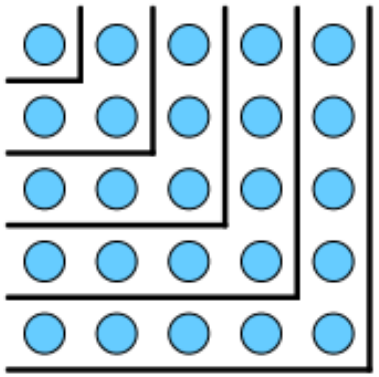


Perfect Squares

If we then group the dots as shown, what do you notice?



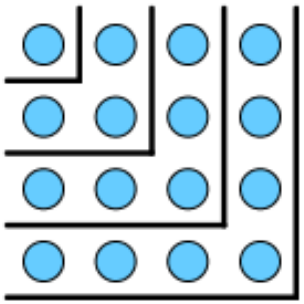
$16 = 4^2$



$25 = 5^2$

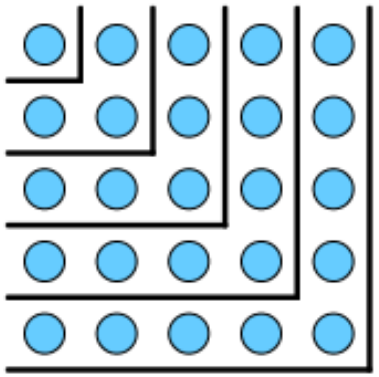
Perfect Squares

If we then group the dots as shown, what do you notice?



$16 = 4^2$

$16 = 1 + 3 + 5 + 7$



$25 = 5^2$

$25 = 1 + 3 + 5 + 7 + 9$

Perfect squares can be built using consecutive odd positive integers.

Perfect Squares

These examples demonstrate the following fun fact:

> The perfect square $s = n^2$ (where n is a positive integer) can be illustrated using dots arranged in an $n \times n$ square grid, and is equal to the sum of the first n consecutive odd positive integers.

This fact can be used to perform efficient calculations.

Example

Question:

What is the sum of the first 12 consecutive odd positive integers?

Solution:

The first 12 consecutive odd positive integers can be illustrated using dots arranged into a 12×12 square grid. Therefore, the sum of these integers is equal to $12^2 = 144$.

Problem Set

1. What is the sum of the first 99 consecutive odd positive integers?
2. If 1225 is the sum of the first m consecutive odd positive integers, what is the value of m?
3. What is the sum of the odd integers from 1 to 50?
4. What is the value of the sum $1 + 3 + 5 + \ldots + 141 + 143 + 145$?
5. What is the value of the sum $17 + 19 + 21 + \ldots + 207 + 209 + 211$?
6. What is the value of the sum $3 + 9 + 15 + \ldots + 423 + 429 + 435$?
7. What is the value of the sum $2 + 4 + 6 + \ldots + 296 + 298 + 300$?