Intermediate Math Circles

October 28, 2020

SHELDON'S SHELLS - SOLUTIONS
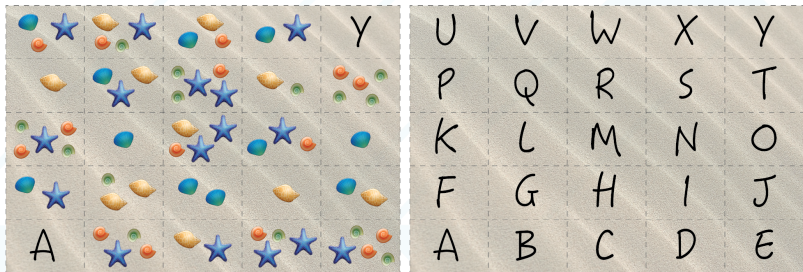
The Centre for Education in Mathematics and Computing
Faculty of Mathematics, University of Waterloo

www.cemc.uwaterloo.ca

Sheldon's Shells

Label the areas of the beach as shown. Recall that Sheldon only moves either up or right on his way from A to Y.

Problem Set #1

Question:

There are three different paths that Sheldon could take from A to the area located one up and two to the right of A (containing two blue shells). What are these three paths? For each of these three paths, how many shells will Sheldon collect, including the two blue shells?

Answer:

Path 1: Taking A F G H, Sheldon will collect 7 shells.

Path 2: Taking A B G H, Sheldon will collect 9 shells.

Path 3: Taking A B C H, Sheldon will collect 8 shells.

Problem Set #2

Question:

Suppose Sheldon walks from A to Y by strictly following the edges of the beach. What is the maximum number of shells that Sheldon could collect?

Answer:

There are two paths Sheldon could take. He could go all the way up and then all the way to the right, collecting 20 shells. Or he could go all the way to the right and then all the way up, collecting 23 shells. Therefore, the maximum number of shells that Sheldon could collect, while strictly following the edges of the beach, is 23.
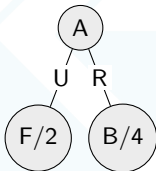
Problem Set #3

Question:

While walking from A to Y, Sheldon stops part way and notices that he has collected exactly 8 shells so far, including the shells in the area in which he has stopped. Identify all the possible areas in which Sheldon may have stopped.

Answer:

One way to solve this question is to build a tree starting with A and illustrating all the possible paths Sheldon could take. Since from A Sheldon can either move up to F (where he collects 2 shells) or right to B (where he collects 4 shells), the tree would begin like this:
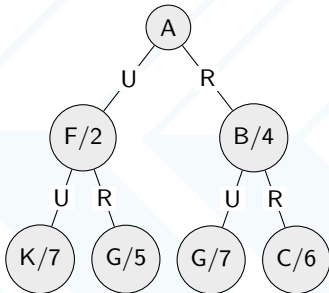
Problem Set #3

Answer Continued:

From F Sheldon can either move up to K (where he collects 5 more shells for a total of 7) or right to G (where he collects 3 more shells for a total of 5). From B Sheldon can either move up to G (where he collects 3 more shells for a total of 7) or right to C (where he collects 2 more shells for a total of 6). Those branches are added to the tree like this:
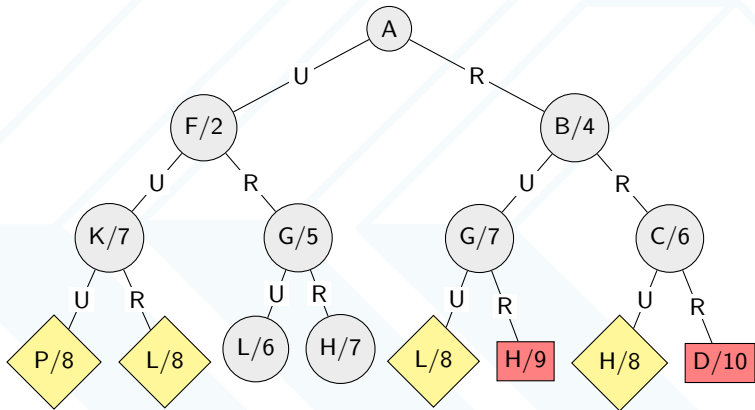
Problem Set #3

Answer Continued:

We can continue to add branches representing Sheldon's possible paths and cumulative amount of shells collected:

Problem Set #3

Answer Continued:

The yellow diamond areas are areas on the path where Sheldon will have collected exactly 8 shells. We do not need to add any new branches from these areas since any further movement will result in collecting more than 8 shells. We also do not need to add any new branches from the red rectangular areas since Sheldon has already exceeded 8 shells along these paths.
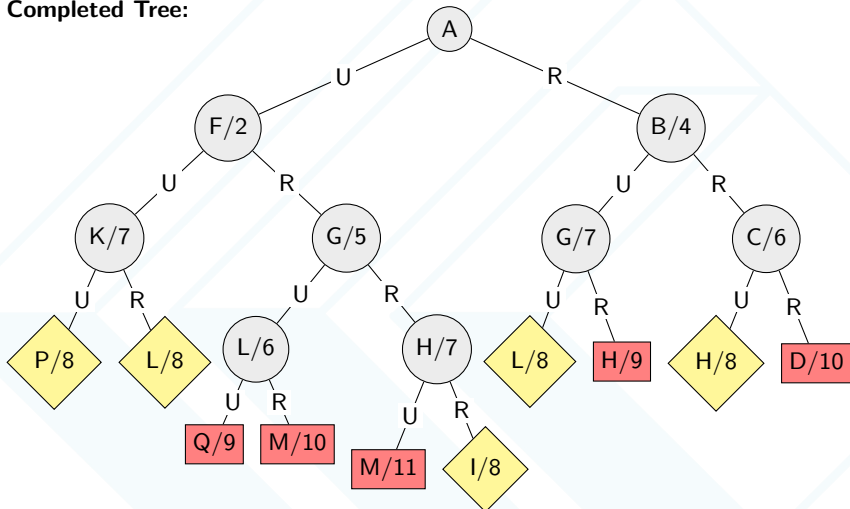
The completed tree is on the next slide.

From the completed tree, we can see that the only areas Sheldon could have stopped in (after collecting exactly 8 shells) are areas P, L, I, and H.

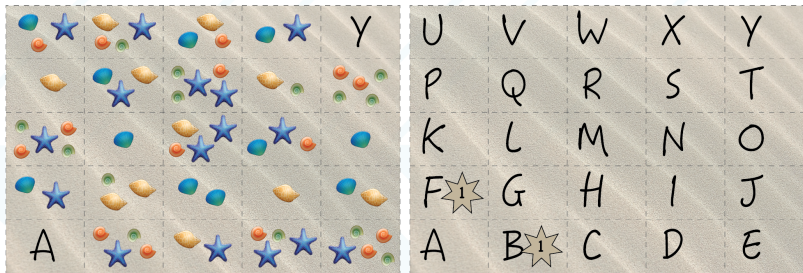# Problem Set #3

Completed Tree:

Problem Set #4

Question:

How many possible paths could Sheldon take from A to Y?

Answer:

For each area we can record how many paths lead to this area. For example, only 1 path leads to F (up) and only 1 path leads to B (right).

| U	V	W	X	Y
P	Q	R	S	T
K	L	M	N	O
F ☆1	G	H	I	J
A	B ☆1	C	D	E

Problem Set #4

Answer Continued:

Only 1 path leads to K (up, up) and only 1 path leads to C (right, right). Since area G can be reached from areas F and B, the number of paths that lead to G is the sum of the number of paths leading to F and leading to B.
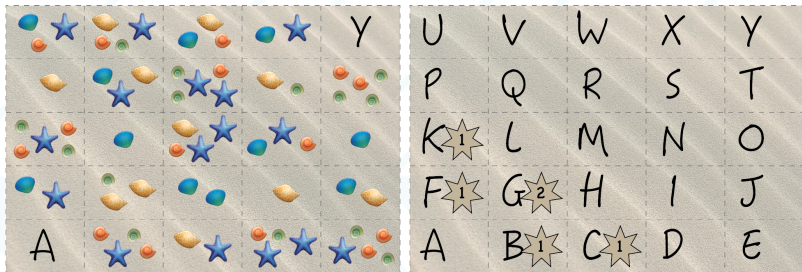
U	V	W	X	Y
P	Q	R	S	T
K (1)	L	M	N	O
F (1)	G (2)	H	I	J
A	B (1)	C (1)	D	E

Problem Set #4

Answer Continued:

Continue filling in the areas by working diagonally up to area Y. The number filled in for area Y will be the number of possible paths Sheldon could take from A to Y.

U 1	V 5	W 15	X 35	Y 70
P 1	Q 4	R 10	S 20	T 35
K 1	L 3	M 6	N 10	O 15
F 1	G 2	H 3	I 4	J 5
A	B 1	C 1	D 1	E 1
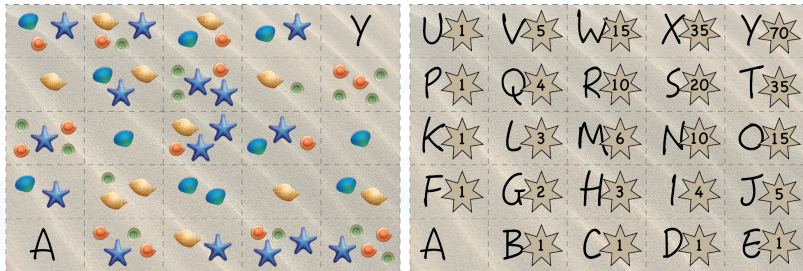
Problem Set #4

Alternate Answer:

To get from A to Y Sheldon must move up four times and move right four times, in some order. How many different ways are there to arrange four U's and four R's?

There are 8 choices for the first move, 7 choices for the next move, 6 choices for the third move, and so on. Therefore, four U's and four R's can be arranged in $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 8!$ (8 factorial) ways. However, these ways are not all different. Can you explain why?

What we really want to do is pick four out of the eight moves and designate them as up. Then the remaining four moves get filled in as right. So we choose 4 from 8.

$$\text{"8 choose 4"} = {_8}C_4 = \binom{8}{4} = \frac{8!}{4!(8-4)!} = \frac{40\,320}{24(24)} = \frac{1680}{24} = 70$$

Problem Set #5

Question:

What is the maximum number of shells that Sheldon could collect while walking from A to Y?

Answer:

One way to figure out the maximum number of shells that Sheldon could collect on his way from A to Y is to continue building a tree until all paths reach Y. Then, the bottom of the tree can be scanned for the largest number.
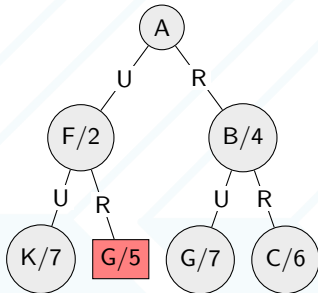
There are ways to simplify the tree as you go, so that you do not need to draw all 70 paths.

Problem Set #5

Answer Continued:

One way to simplify the tree is to *abandon* a path when a better path is found. For example, consider the beginning of the tree:


There are two ways to get to area G. Taking the path $A\ B\ G$ is better since it results in collecting more shells. We can abandon the inferior path $A\ F\ G$ by not adding any more branches to it.

Problem Set #5

Answer Continued:

Another way to simplify the tree is to *merge* paths that are equally good. For example, path $A\ F\ K\ L$ and path $A\ B\ G\ L$ both result in Sheldon being in area L with a total of 8 shells. We can treat these two paths as one.
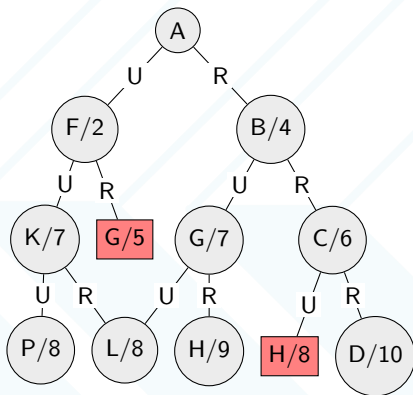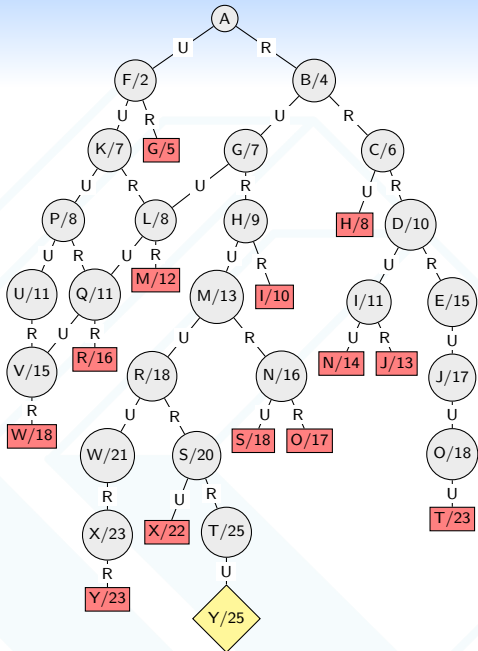
Problem Set #5

Answer Continued:

Here is the completed tree.

The maximum number of shells that Sheldon could collect while walking from A to Y is 25.

Problem Set #5

Alternate Answer:

Another way to approach this problem is to make the following observation:

> The maximum number of shells that can be collected by Sheldon upon reaching *any* area, is equal to the number of shells in that area *plus* . . .
>
> - the total number of shells collected so far if approaching from the left, or
> - the total number of shells collected so far if approaching from below.
>
> We take the greater of these two options.

Problem Set #5

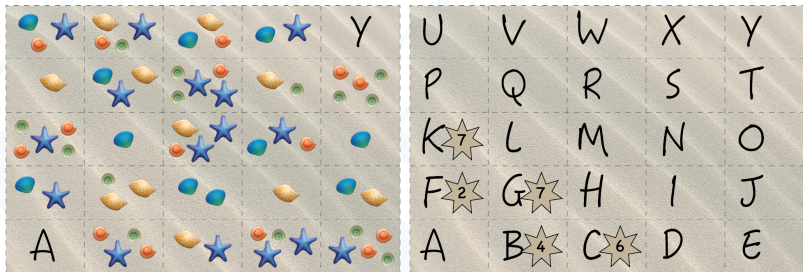**Alternate Answer Continued:**

For example, the maximum number of shells upon reaching area F is $2 + 0 = 2$ since there are 2 shells in area F and the only option is to approach from below (bringing 0 shells with you). The maximum number of shells upon reaching area B is $4 + 0 = 4$ since there are 4 shells in area B and the only option is to approach from the left (bringing 0 shells with you).
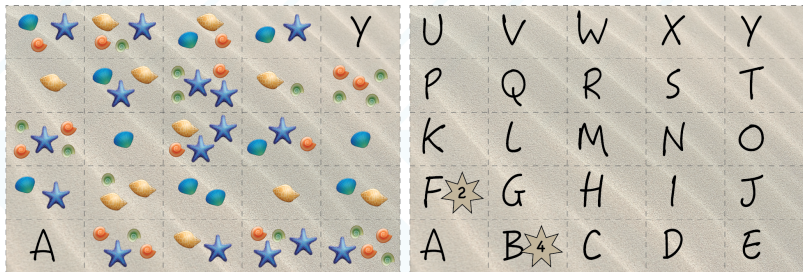
Problem Set #5

Alternate Answer Continued:

Similarly, the maximum number of shells upon reaching area K is $5 + 2 = 7$ and the maximum number of shells upon reaching area C is $2 + 4 = 6$. Area G is a little more interesting. There are 3 shells in area G and the best option is to approach from below (bringing 4 shells with you). Therefore, the maximum number of shells upon reaching area G is $3 + 4 = 7$.

Problem Set #5

Alternate Answer Continued:

Continue filling in the areas by working diagonally up to area Y. The number filled in for area Y will be the maximum number of shells that Sheldon could collect while walking from A to Y.

				Y
A				

U 11	V 15	W 21	X 23	Y 25
P 8	Q 11	R 18	S 20	T 25
K 7	L 8	M 13	N 16	O 18
F 2	G 7	H 9	I 11	J 17
A	B 4	C 6	D 10	E 15