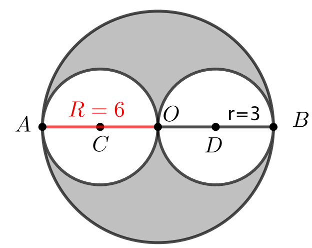
Intermediate Math Circles
Circles and Area
Problem Set Solutions

Exercise 1 Solution

We will find the area of the shaded region by subtracting the areas of the smaller circles from the area of the larger circle. Let R be the radius of the larger circle and r be the radii of the smaller circles.



Since the diameter of each of the smaller circles is the radius of the larger circle, we have $R = \frac{12}{2} = 6$ and $r = \frac{6}{2} = 3$.

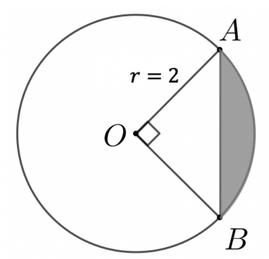
$A_{\text{larger}} = \pi R^2 = \pi(6)^2 = 36\pi$
$A_{\text{smaller}} = \pi r^2 = \pi(3)^2 = 9\pi$

Therefore, the area of the shaded region is $36\pi - 2(9\pi) = 18\pi$.

Exercise 2 Solution

To find the area of the shaded region, we will find the area of the sector of the circle with arc AB and subtract the area of $\triangle AOB$. Note that the triangle is a right isosceles triangle and therefore, the base and height are both equal to the radius which is 2.

$$A_{wholeCircle} = \pi r^2 = \pi(2)^2 = 4\pi$$
$$A_{sector} = \left(\frac{90}{360}\right)4\pi = \pi$$
$$A_{triangle} = \frac{bh}{2} = \frac{(2)(2)}{2} = 2$$



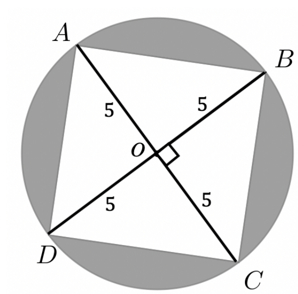
Therefore, the area of the shaded region is $\pi - 2$.

Exercise 3 Solution

There are a few different ways to approach this problem. We will outline two approaches. Each of these approaches relies upon the following facts that we will not prove:

1) The two diagonals of the inscribed square intersect at the centre, O, of the circle.

2) The two diagonals of the inscribed square bisect each other and meet at right angles.



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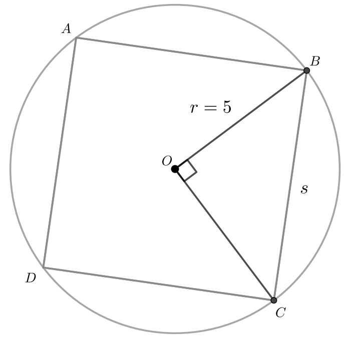
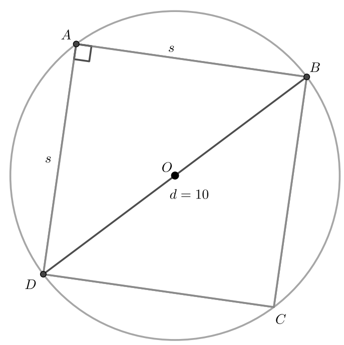
Approach 1: Recognize that the shaded region in this problem consists of four identical shaded regions, each having an area that can be calculated by subtracting the area of a triangle from the area of a sector of a the circle (as we saw in Exercise 2).
The final calculation is as follows: Area $= 4\left(\frac{\pi(5)^2}{4} - \frac{5^2}{2}\right) = 25\pi - 50$.
Approach 2: Recognize that the area of the shaded region is the area of the circle minus the area of the inscribed square.
The area of the circle $\pi r^2 = \pi(5)^2 = 25\pi$.
Let s be the side length of the square as shown in the figures below.
Note that $\triangle BOC$ is a right isosceles triangle. Therefore, its base and height are both equal to the radius which is $r = 5$. We can calculate the value of s^2
as follows:
Using the Pythagorean Theorem on $\triangle BOC$, we get

$$\begin{aligned} s^2 &= r^2 + r^2 \\ s^2 &= (5)^2 + (5)^2 \\ s^2 &= 25 + 25 \\ s^2 &= 50 \end{aligned}$$



Alternatively, using the Pythagorean Theorem on $\triangle BAD$, with diameter $d = 10$, we get

$$\begin{aligned} d^2 &= s^2 + s^2 \\ 10^2 &= 2s^2 \\ 100 &= 2s^2 \\ 50 &= s^2 \end{aligned}$$

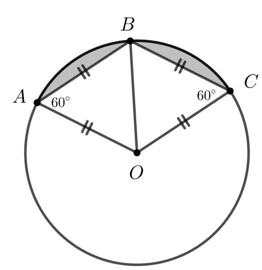


Each of these calculations tells us that the area of the square is $s^2 = 50$.
Therefore, the area of the shaded region is $25\pi - 50$.

Exercise 4 Solution

Solution 1

Since $OABC$ is a rhombus it must also be a parallelogram. Therefore $\angle AOC + \angle OAB = 180°$. Since $\angle AOC = 120°$ then $\angle OAB = 60°$.
Similarly, we can show that $\angle OCB = 60°$.
Construct line segment OB.
We will now show $\triangle AOB$ is equilateral.



Note that the $\triangle AOB$ is an isosceles triangle with sides $AO = AB$ Therefore, $\angle AOB = \angle OBA$. We also know that the sum of the angles in a triangle is 180°. Therefore:

$$\begin{aligned}\angle AOB + \angle OBA + \angle OAB &= 180\\ \angle AOB + \angle AOB + 60 &= 180\\ 2\angle AOB &= 120\\ \angle AOB &= 60\end{aligned}$$

Therefore, $\angle AOB = \angle OBA = \angle OAB = 60°$ and $\triangle AOB$ is an equilateral triangle with a side length of 10. Similarly, $\triangle BOC$ is an equilateral triangle with a side length of 10.

(Note: For this specific example, there is another way to show that $\triangle AOB$ is an equilateral triangle. We know that OB is a radius but so is OA. Therefore $OB = OA$. Also since they are sides of a rhombus then $OA = AB$. Therefore, $OB = OA = AB$ and hence, $\triangle AOB$ is an equilateral triangle with side length of 10.)

We will look at the shaded region on sector AOB. To find the area of this shaded region, we will take the area of the sector and subtract the area of $\triangle AOB$.

$A_{wholeCircle} = \pi r^2 = \pi(10)^2 = 100\pi$
$A_{sector} = \left(\frac{60}{360}\right)100\pi = \frac{50\pi}{3}$
$A_{equilateraltriangle}$
$= \frac{\sqrt{3}s^2}{4} = \frac{\sqrt{3}(10)^2}{4} = 25\sqrt{3}$

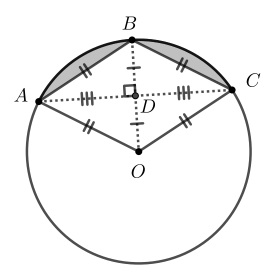
Therefore, the area of the shaded region is $\dfrac{50\pi}{3} - 25\sqrt{3}$.

similarly, the area of the shady region on sector BOC is also $\dfrac{50\pi}{3} - 25\sqrt{3}$.

Total area of the shaded region is $2(\dfrac{50\pi}{3} - 25\sqrt{3}) = \dfrac{100\pi}{3} - 50\sqrt{3}$.

Solution 2

We are going to use the following properties of a rhombus: the diagonals of a rhombus bisects each other and meet at a right angle.
To start this solution we will draw in the two diagonals and label the appropriate equal lengths and the right angle. This is shown to the right.



In $\triangle ABD$, we know that $DB = \frac{1}{2}OB$. Since OB is a radius, $DB = \frac{1}{2}OB = \frac{1}{2}(10) = 5$. We also know that $AB = 10$ and $\angle ADB = 90°$.

Therefore, using the Pythagorean Theorem,

$$\begin{aligned} AD^2 &= AB^2 - BD^2 \\ &= (10)^2 - (5)^2 \\ &= 100 - 25 \\ AD^2 &= 75 \\ AD &= \sqrt{75}, \text{ since } AD > 0. \end{aligned}$$

Therefore, the area of $\triangle ABD = \frac{bh}{2} = \frac{5\sqrt{75}}{2}$.
Similarly, we can show that the areas of $\triangle AOD, \triangle COD$ and $\triangle CBD$ are each equal to $\frac{5\sqrt{75}}{2}$.
Since the area of the rhombus equals the sum of the 4 triangle, the area of the rhombus is $4\left(\frac{5\sqrt{75}}{2}\right) = 10\sqrt{75}$.

Now to find the area of the shaded region we need to find the area of the sector and subtract the area of the rhombus.

$A_{wholeCircle} = \pi r^2 = \pi(10)^2 = 100\pi$
$A_{sector} = \left(\frac{120}{360}\right) 100\pi = \frac{100\pi}{3}$
$A_{rhombus} = 10\sqrt{75}$

Therefore, the area of the shaded region is $\dfrac{100\pi}{3} - 10\sqrt{75}$.

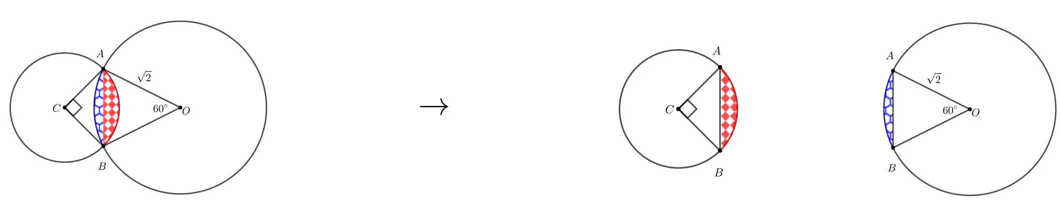
NOTE:
This answer looks different than Solution 1. There is a way to simplify the second solution.

We can rewrite $\sqrt{75}$ as $\sqrt{25}\sqrt{3} = 5\sqrt{3}$.

Now the area in Solution 2 becomes $\dfrac{100\pi}{3} - 10\sqrt{75} = \dfrac{100\pi}{3} - 10(5\sqrt{3}) = \dfrac{100\pi}{3} - 50\sqrt{3}$.

Exercise 5 Solution

When we separate the two circles we get the following:



Note, when separated, in the circle centred at O, that the $\triangle AOB$ is an isosceles triangle with sides $AO = OB$ and $\angle BOA = 60°$ Therefore, $\angle OAB = \angle OBA$. Therefore, from previous examples we know that $\triangle AOB$ is equilateral and $AB = \sqrt{2}$.

Now, in the circle centred at C, we know $AB = \sqrt{2}$ and $\triangle ACB$ is a right isosceles triangle.

Using the Pythagorean Theorem on $\triangle ACB$, we get

$$\begin{aligned}(\sqrt{2})^2 &= r^2 + r^2\\ 2 &= 2r^2\\ 1 &= r^2\\ r &= 1,\ \text{since } r > 1\end{aligned}$$

To get the total shaded area, we now need to find the area of two shaded regions.

1. The first is the area of the shaded region in the circle centred at O.

   $A_{wholeCircle} = \pi r^2 = \pi(\sqrt{2})^2 = 2\pi$
   $A_{sector} = \left(\frac{60}{360}\right) 2\pi = \frac{\pi}{3}$
   $A_{equilateraltriangle}$
   $= \frac{\sqrt{3}s^2}{4} = \frac{\sqrt{3}(\sqrt{2})^2}{4} = \frac{\sqrt{3}}{2}$

   Therefore, the area of this shaded region is $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$.

2. The second area is the area of the shaded region in the circle centred at C.

   $A_{wholeCircle} = \pi r^2 = \pi(\sqrt{1}2)^2 = \pi$
   $A_{sector} = \left(\frac{90}{360}\right) \pi = \frac{\pi}{4}$
   $A_{triangle} = \frac{bh}{2} = \frac{(1)(1)}{2} = \frac{1}{2}$

   Therefore, the area of this shaded region is $\frac{\pi}{4} - \frac{1}{2}$

Therefore, the total area of the shaded region is

$$\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) + \left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{7\pi}{12} - \frac{\sqrt{3}}{2} - \frac{1}{2}.$$