Intermediate Math Circles
Addition Magician
Problem Set Solution

Exercise 1: What would a winning strategy be if the game is played to a total of 55?

Solution 1:
The winning total is 55 which is a multiple of 11. Therefore, there is a change in strategy.

A winning strategy in this variation is to go second and, on each turn, if the other player adds n, then add $11 - n$, so that the total changes by 11 in total over the two turns. For example, if they add 4 then you(the second player) add 7. This way the second player will bring the total to 11, 22, 33, 44, and then 55 to win.

Exercise 2: What would a winning strategy be if the players could chose from 1 to 15 and play to a total of 300?

Solution 2:
The player that brings the total to a number between 285 and 299 inclusive will generally lose the game since the next player can reach 300. Since the allowable numbers in this variation are 1 to 15, we focus on multiples of 16. Since 300 is 12 more than a multiple of 16, the winning strategy is to go first and start with 12. Then, if the other player adds n, you add $16 - n$, so that the total changes by 16 over the two turns. For example, if they add 7, you add 9. The first player can always bring the total to the next number that is 12 more than a multiple of 16, eventually reaching 300 to win.

Exercise 3: What would a winning strategy be if the players are allowed to use the numbers from 1 to n, with $n > 1$, but must play to a total of T where T is some positive integer larger than $3n$?

Solution 3:
We need to consider different cases for T:

If T is a multiple of $n + 1$, then go second. Whatever number the other player chooses, you choose the number that totals $n + 1$ when summed with their chosen number. This means you will bring the total to each multiple of $n + 1$, in turn, eventually reaching T. (Exercise 1 is an instance of this case.)

If T is not a multiple of $n + 1$, then go first. Find the remainder when T is divided by $n + 1$ and start with this number. Whatever number the other player chooses, choose the number that totals $n + 1$ when summed with their chosen number. Eventually you will bring the total to T. (Exercise 2 is an instance of this case.)

Exercise 4: Another variation: Start at 52 and on your turn you can subtract 1, 2, 3, 4, 5, 6, 7, 8, 9, or 10. The first person to 0 wins the game. What be a winning strategy now?

Solution 4:
You likely noticed that the player that brings the total to 1, 2, 3, 4, 5, 6, 7, 8, 9, or 10 generally loses the game on the next turn. The next player can reach 0 by subtracting 1, 2, 3, 4, 5, 6, 7, 8, 9, or 10, respectively, and so will win the game as long as they choose the correct number. Therefore, the player that brings the total to 11 is guaranteed to be able to bring the total to 0 on their next turn.

Using similar reasoning, the player that brings the total to 22 is guaranteed to be able to bring the total to 11 on their next turn. Also the player that brings the total to 33 is guaranteed to be able to bring the total to 22 on their next turn, and the player that brings the total to 44 is guaranteed to be able to bring the total to 33 on their next turn.

So now the target numbers are 44, 33, 22, and 11. We can describe the strategy more concisely as follows: Go first and start by subtracting 8. For all turns that follow, if the other player subtracts n, then subtract $11 - n$.

Exercise 5: Here is a similar game

Careful Clipping

You Will Need:

- Two players
- 10 paper clips
  (or other small objects)

How to Play:

1. Start with a pile of 10 paper clips.
2. Players alternate turns.
3. On your turn, you can remove 1, 2 or 3 paper clips from the pile.
4. The player who removes the last paper clip, *loses*.

Can you determine a winning strategy for this game?

Solution 5:
Let the two players be Player 1 and Player 2.

You likely noticed that the player that brings the number of paper clips in the pile to 1 is guaranteed to win the game, and the player that brings the number of paper clips in the pile to 2, 3, or 4 generally loses the game. This is because the next player can bring the pile to 1 paper clip by removing 1, 2 or 3 paper clips, respectively.

Using similar reasoning, we can show that the player that brings the number of paper clips to 5 is guaranteed to be able to bring the number to 1 on their next turn, and the player that brings the number of paper clips to 9 is guaranteed to be able to bring the number to 5 on their next turn. This means that Player 1 has a winning strategy for this game and it goes as follows:

Start by removing 1 paper clip, reducing the total number of paper clips to 9. On your next turn, remove the required number of paper clips are needed to bring the total to 5. On your turn after that, remove the required number of paper clips are needed to bring the total to 1. (Our analysis above explains why each of these moves will be possible within the rules of the game.)

Notice that the "target numbers" (9, 5, and 1) all differ by 4. We can instead describe the strategy as follows: Go first and start by removing 1 paper clip. For all turns that follow, if the other player has just removed n paper clips, then remove $4 - n$ paper clips. (These two turns, combined, will reduce the number of paper clips by 4.)

We are going to give the solutions to 6 and 7 together.

Exercise 6: Here is a variation of Careful Clipping:

- The game is won (instead of lost) by the player who removes the last paper clip.
- Each player can take 1,3 or 4 paper clips.

Can you determine a winning strategy for this game?

Exercise 7: Here is a second variation of Careful Clipping:

- The game is played with 14 paper clips.
- The game is won (instead of lost) by the player who removes the last paper clip.
- Each player can take 1,3 or 4 paper clips.

Can you determine a winning strategy for this game?

Solution 6 and 7:
For exercise 6, the first player will have a winning strategy. For exercise 7, the second player will have a winning strategy.

The winning strategy for these questions is dependant on the starting number of paper clips. On the next page, we present a strategy starting with 1 to 14 paper clips.

Let the two players be Ally and Bri. In each game, Ally will go first.

Starting Pile	Winner	Reasoning
1	Player 1	Ally takes the clip and wins.
2	Player 2	Ally must take 1 clip. Bri takes the remaining clip and wins.
3	Player 1	Ally takes all 3 clips and wins.
4	Player 1	Ally takes all 4 clips and wins.
5	Player 1	Ally takes 3 clips, leaving 2 clips for Bri's turn. As above, if a pile has 2 clips, the second player will win. Since it is Bri's turn, the second player (starting from 2 clips) is Ally (Player 1).
6	Player 1	Ally takes 4 clips, leaving 2 clips for Bri's turn. As above, if a pile has 2 clips, the second player will win. Since it is Bri's turn, the second player (starting from 2 clips) is Ally (Player 1).
7	Player 2	Ally must take 1, 3, or 4 clips, leaving 6, 4, or 3 clips for Bri's turn. As above, if a pile has 6, 4, or 3 clips, the first player will win. Since it is Bri's turn, the first player is Bri (Player 2).
8	Player 1	Ally takes 1 clip, leaving 7 clips for Bri's turn. As above, if a pile has 7 clips, the second player will win. Since it is Bri's turn, the second player is Ally (Player 1).
9	Player 2	Ally must take 1, 3, or 4 clips, leaving 8, 6, or 5 clips for Bri's turn. As above, if a pile has 8, 6, or 5 clips, the first player will win. Since it is Bri's turn, the first player is Bri (Player 2).
10	Player 1	Ally takes 1 clip, leaving 9 clips for Bri's turn. As above, if a pile has 9 clips, the second player will win. Since it is Bri's turn, the second player is Ally (Player 1).
11	Player 1	Ally takes 4 clips, leaving 7 clips for Bri's turn. As above, if a pile has 7 clips, the second player will win. Since it is Bri's turn, the second player is Ally (Player 1).
12	Player 1	Ally takes 3 clips, leaving 9 clips for Bri's turn. As above, if a pile has 9 clips, the second player will win. Since it is Bri's turn, the second player is Ally (Player 1).
13	Player 1	Ally takes 4 clips, leaving 9 clips for Bri's turn. As above, if a pile has 9 clips, the second player will win. Since it is Bri's turn, the second player is Ally (Player 1).
14	Player 2	Ally must take 1, 3, or 4 clips, leaving 13, 11, or 10 clips for Bri's turn. As above, if a pile has 13, 11, or 10 clips, the first player will win. Since it is Bri's turn, the first player is Bri (Player 2).