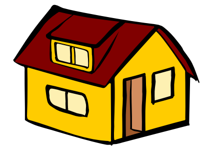
CEMC Math Circles - Grade 11/12

November 25 - December 2, 2020

Dwellings with many Doors - Solutions

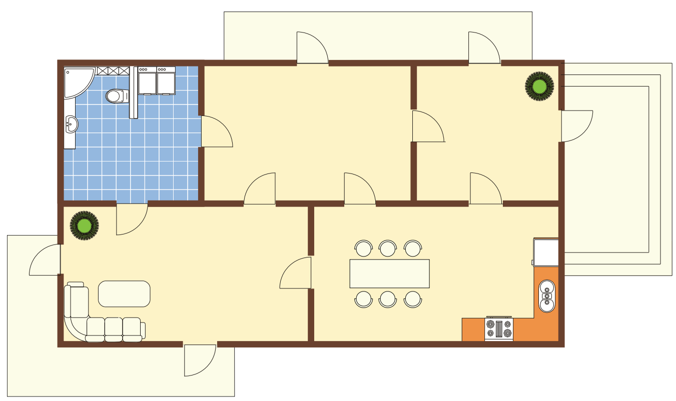


Here are the solutions to this weeks problems. There is a description of each problem, followed by a discussion about the solution.

Problem 1. You live in a large house with several rooms– the floor plan is illustrated below. One day, in the midst of your infinite boredom, you wonder about the following question:

Can you walk through your house in such a way that you go through each door *exactly once*? If so, how many different ways are there to do this?

Remarks: This isn't a trick question. The gaps in the walls are doors, and you can pass through each door exactly once. You can start and end wherever you would like.

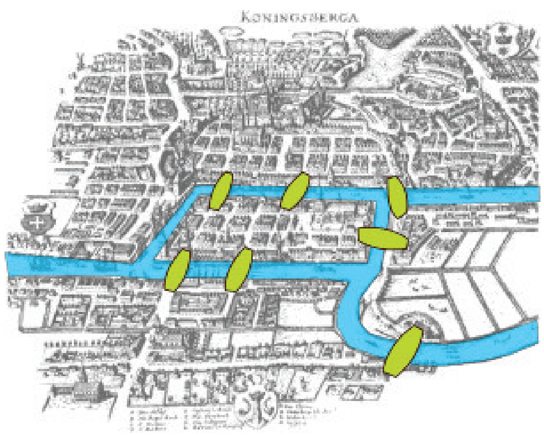


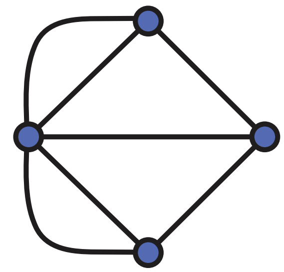
Solution. Here is a very short reason that this is impossible. There are two possibilities for such a walk: either you start and end in the same room, or you start and end in *different* rooms (we'll consider the outside of the house to be a "room"). Notice that if you start and end in the same room, this means that there must be an *even* number of doors leading in or out of each room– because no matter what, you have to leave every time you enter. Similarly, if you start and end in different rooms, this means that *exactly two* of the rooms have an odd number of doors leading in or out, and the rest must have an even number.

Looking at the schematic we see that in fact, *four* of the rooms (don't forget the outside of the house) have an odd number of doors, and so we conclude that no such walk is possible! □

This kind of problem was historically very interesting, and the most google-able version is called the *Bridges of Königsberg* problem. You can find lots of information on the Wikipedia page here. It was solved by Euler in 1736, and laid the foundations of topology and graph theory.

The problem is the following (it's very similar to Problem 1): can you walk through the city of Königsberg (shown below) in such a way that you cross each bridge exactly once?





Euler pointed out that really, the drawing is irrelevant; what matters is how the different land masses are connected to each other! He drew a "graph," which consists of a point for each land mass, and and edge whenever two land masses were connected by a bridge. Now, the question really asks: can you walk through this graph in such a way that you traverse each edge exactly once?

In modern language, an **Eulerian circuit** for a graph is a walk where you start and end at the same place, and an **Eulerian path** is a walk where you start and end in different places. Euler proved (in fact, it's the same proof that we gave for Problem 1!) that a graph has an Eulerian circuit exactly when each node has an even number of edges, and that a graph has an Eulerian path exactly when there are exactly two nodes with an odd number of edges. It sounds fancy, but it's the exact same analysis that we did for Problem 1. In particular, you cannot walk through the city of Königsberg in such a way, since there are four nodes with an odd number of edges. This is a very useful trick, and we'll apply it to solve Problem 2.

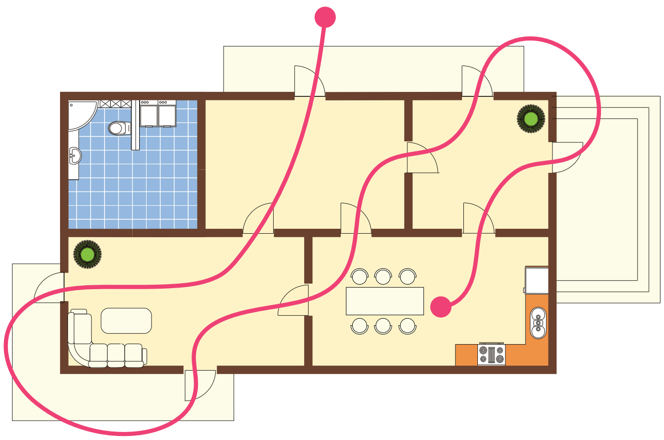
Problem 2. Your butler becomes tired of you endlessly wandering around the house, and shuts the doors to the bathroom (the room in the top left corner) Does this help matters? In other words, can you now walk through the house in such a way that you go through each remaining door exactly once? What if you insist on starting and ending at the same place?

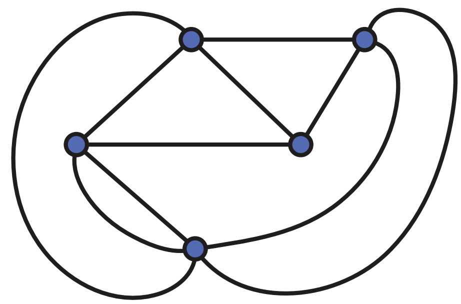
If you can do either of these walks, how many different walks of each kind are there?

What is the smallest number of doors that you can close so that you can walk exactly once through each remaining door, and start and end in the same place?

Solution. We analyze the new floor plan in the same way. Now, exactly two of the rooms have an odd number of doors.

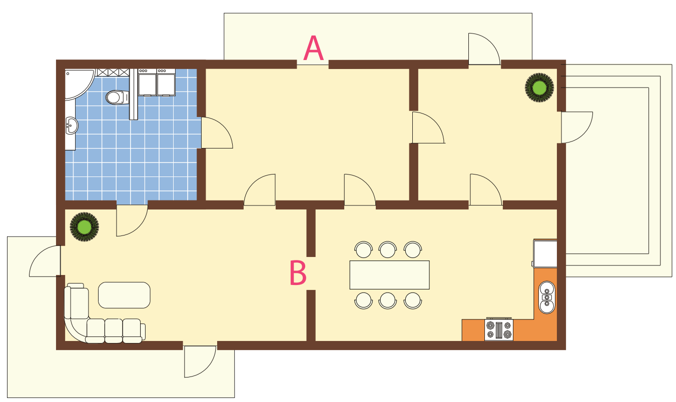
That means that there is a Eulerian path (where we start and end in different rooms– one such path is illustrated in pink) but there *cannot* be an Eulerian circuit (where we start and end in





the same room). We can also draw the corresponding graph! We draw a node for each room, and connect two nodes with an edge if they share a door.

There are a lot of possible paths! If you think you have a way of counting them all, let me know. If you want to be able to find an Eulerian circuit, your butler will have to close different doors. You will need to close at least two– think about why! However, you can do it with only two. For example, if you close the doors labelled A and B, you will be able to find an Eulerian circuit.



If you were able to do both of these problems and want something interesting to think about, try this challenge problem: how many ways are there to close two doors, so that the resulting house admits an Eulerian circuit?

□