CEMC Math Circles - Grade 11/12

February 24 - March 2, 2021
It's True Because it Isn't False!

Prove each of the following statements using a proof by contradiction approach.

1. There do not exist integers x and y such that $10x - 25y = 6$.

   Proof:

   Suppose, for a contradiction, that there do exist integers x and y such that $10x - 25y = 6$.

   Since x and y are integers, the quantity $10x - 25y$ is an integer as well.
   Since $10x - 25y = 5(2x - 5y)$, the integer 5 must be a factor of the integer $10x - 25y$.
   Since $10x - 25y = 6$, this means the integer 5 must also be a factor of 6.
   This is a contradiction because 5 is not a factor of 6.

   Therefore, we conclude that there cannot exist integers x and y such that $10x - 25y = 6$.

2. If x and y are positive real numbers, then $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$.

   Proof:

   Suppose, for a contradiction, that there are positive real numbers x and y for which $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.

   Squaring both sides of the equation gives

$$\begin{aligned}
\left(\sqrt{x+y}\right)^2 &= \left(\sqrt{x} + \sqrt{y}\right)^2 \\
x + y &= \left(\sqrt{x}\right)^2 + 2\sqrt{x}\sqrt{y} + \left(\sqrt{y}\right)^2 \\
x + y &= x + 2\sqrt{x}\sqrt{y} + y \\
0 &= 2\sqrt{xy} && \text{since } x, y > 0 \\
0 &= \sqrt{xy} \\
0 &= xy
\end{aligned}$$

   Since $xy = 0$ we must have $x = 0$ or $y = 0$. This is a contradiction. Since x and y are both positive real numbers, it cannot be the case that $x = 0$ and it cannot be the case that $y = 0$.

   This means there cannot be positive real numbers x and y for which $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.

   Therefore, we conclude that if x and y are positive real numbers, then we must have $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$.

3. Extra Challenge: If the parabola $y = ax^2 + bx + c$ (with a, b, c non-zero real numbers) touches or crosses the x-axis, then a, b, c cannot form a geometric sequence, in that order.

   Solution:

   Let $y = ax^2 + bx + c$ be a parabola, with a, b, c non-zero real numbers, that crosses or touches the x-axis. We want to prove that a, b, c cannot form a geometric sequence, in that order.

   Suppose, for a contradiction, that a, b, c is a geometric sequence, in that order.

   Since a, b, c is a geometric sequence, it has a common ratio, r, such that $b = ar$ and $c = ar^2$.
   Since we know that $b \neq 0$, we know that $r \neq 0$.

Since the parabola crosses or touches the x-axis, the quadratic equation $ax^2 + bx + c = 0$ has at least one real solution. So the discriminant of the quadratic must be at least zero.

Since the discriminant is

$$b^2 - 4ac = (ar)^2 - 4a(ar^2) = (ar)^2 - 4(ar)^2 = -3(ar)^2$$

we must have $-3(ar)^2 \geq 0$.

However, since $a \neq 0$ and $r \neq 0$, we must have $ar \neq 0$. It follows that $(ar)^2 > 0$ and so $-3(ar)^2 < 0$. This is a contradiction since we cannot have both $-3(ar)^2 \geq 0$ and $-3(ar)^2 < 0$.

Therefore, it cannot be the case that a, b, c is a geometric sequence, in that order.

Discussion: Proving statements can be one of the most rewarding parts of mathematics, but it can also be challenging. There are a variety of proof techniques that we can use, one of which is proof by contradiction. While there are no fixed rules about which technique to use to prove a statement (one of the reasons why writing a proof can be challenging!), there are some statements that lend themselves well to a proof by contradiction approach. The statements from this resource fall into this category.

Our first problem here involved showing that a pair of integers with some property *cannot* exist, and the third problem here involved showing that a triple of real numbers *cannot* satisfy a certain condition. Taking a proof by contradiction approach allowed us to see what would happen if the pair of integers *did* exist and if the real numbers *did* satisfy the condition. We were able to do some algebra and, quite quickly, we discovered that these situations lead to contradictions and so we could rule them out.

Think about how you would prove these statements without taking a proof by contradiction approach. For example, for the first statement, you would need to argue directly that every possible pair of integers fails to satisfy the equality. How would you make this argument? Sometimes, it can be challenging to argue that objects *do not* exist or *do not* satisfy a certain condition *directly* (especially when there are infinitely many objects to rule out!). You may not need to do a proof by contradiction, but thinking about the problem in this way will often lead you to a nice argument.

In contrast, you might find it very natural to prove the second statement here without using a proof by contradiction approach. Given two positive real numbers x and y, can you check directly that $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$?

Here is an example of a *direct proof of statement 2:*

Let x and y be positive real numbers. Then we have

$$(\sqrt{x+y})^2 - (\sqrt{x} + \sqrt{y})^2 = (x+y) - (x + 2\sqrt{x}\sqrt{y} + y) = -2\sqrt{x}\sqrt{y}$$

Since $\sqrt{x} > 0$ and $\sqrt{y} > 0$ we have $\sqrt{x}\sqrt{y} > 0$ and so $-2\sqrt{x}\sqrt{y} < 0$. It follows that

$$(\sqrt{x+y})^2 - (\sqrt{x} + \sqrt{y})^2 < 0$$

and hence

$$(\sqrt{x+y})^2 < (\sqrt{x} + \sqrt{y})^2$$

Since $\sqrt{x+y} > 0$ and $\sqrt{x} + \sqrt{y} > 0$, it must be the case that $\sqrt{x+y} < \sqrt{x} + \sqrt{y}$. In particular, this means that $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$ as desired.