CEMC Math Circles - Grade 11/12

April 7, 2020 - April 13, 2020

Silly Square Roots – Solutions

Here we expand on our use of writing Python programs to solve Question 5a on the 2019 Euclid Contest and two variations of this problem.

Problem 1

Determine the two pairs of positive integers (a, b) with $a < b$ that satisfy the equation $\sqrt{a}+\sqrt{b} = \sqrt{50}$.

Discussion of Programming Solution

We saw that the following program gives the two correct answers.

```
import math
for a in range(1, 50):
    for b in range(a + 1, 50):
        if (math.sqrt(a) + math.sqrt(b)) == math.sqrt(50):
            print((a,b))
```

Here are some observations about the mathematics used to inform and write the code:

- The smallest value to consider for a is 1 because it must be a positive integer.
- The smallest value to consider for b is $a + 1$ because we want $a < b$ and both a and b must be integers.
- If $a \geq 50$, then $\sqrt{a} + \sqrt{b} \geq \sqrt{50} + \sqrt{b} \geq \sqrt{50}$. This means we don't need to consider values of a that are greater than or equal to 50. The same applies for b.
- Since we want $a < b$, we have $2\sqrt{a} = \sqrt{a} + \sqrt{a} < \sqrt{a} + \sqrt{b}$. So if $\sqrt{a} + \sqrt{b} = \sqrt{50}$, we get $2\sqrt{a} < \sqrt{50}$. Squaring both sides and rearranging gives $a < \frac{50}{4} < 13$. This means that we could have replaced `range(1,50)` with `range (1,13)`, and hence checked fewer pairs.

Problem 2

Determine the two pairs of positive integers (a, b) with $a < b$ that satisfy the equation $\sqrt{a}+\sqrt{b} = \sqrt{75}$.

Programming Solution

We noticed that changing 50 to 75 in the programming solution to Problem 1, does not produce the correct answer for Problem 2. In this sense, our solution to Problem 1 was "lucky". One approach that does give us the right answer is the following:

```
import math
for a in range(1,75):
    for b in range(a+1,75):
        if abs(math.sqrt(a) + math.sqrt(b) - math.sqrt(75)) < 0.0001:
            print((a,b))
```

Since the `math.sqrt` function gives close approximate values, we can test if $\sqrt{a}+\sqrt{b}$ is approximately equal to $\sqrt{75}$. To do this, we look at the positive difference of these two values. This is called the absolute value and is written $|\sqrt{a}+\sqrt{b}-\sqrt{75}|$ in mathematics and computed using

```
abs((math.sqrt(a) + math.sqrt(b)) - math.sqrt(75))
```

in Python. It turns out that being within 0.0001 is "close enough". That is, if we use Python to search for pairs (a, b) satisfying $|\sqrt{a}+\sqrt{b}-\sqrt{75}| < 0.0001$ as outlined earlier, then we happen to find exactly two pairs: $(3, 48)$ and $(12, 27)$. We were told that there are exactly two solutions to the equation, and so we can be sure that our program has found all of the solutions. On the other hand, if we had used Python to find pairs (a, b) satisfying $|\sqrt{a}+\sqrt{b}-\sqrt{75}| < 0.001$, then we would have instead found three pairs: the two solutions as well as one extraneous pair, $(8, 34)$. If we changed the bound to 0.01 or 0.1, then the true solutions would be hidden among an even larger number of extraneous pairs. You can explore the number of pairs in these cases on your own.
How would you use the results of these searches to help determine the complete set of solutions if you were not told in advance that there were exactly two solutions?

Problem 3

Determine all pairs of positive integers (a, b) with $a < b$ that satisfy the equation $\sqrt{a}+\sqrt{b}=\sqrt{147}$.

Programming Solution

Here we do not know how many pairs of positive integers to look for. A solution to this problem uses a nice combination of mathematics and computer science.

Consider positive integers a and b and suppose that

$$\sqrt{a}+\sqrt{b}=\sqrt{147}.$$

By squaring both sides we get

$$a+2\sqrt{ab}+b=147.$$

Rearranging gives

$$2\sqrt{ab}=147-a-b.$$

Squaring both sides again yields

$$4ab=(147-a-b)^2.$$

This is an equation that does not involve square roots. But we do have to be a bit careful because squaring equations can introduce solutions. Now, we are only considering cases where a and b are positive, so the first time we squared both sides of the equation above did not introduce solutions. It is also true that the solutions we are looking for must satisfy $a+b<147$ (try to show this!) which means the second time we squared both sides of the equation also did not introduce solutions. Now we can test values for a and b that satisfy this equation only using integer operations, which give exact values in Python:

```
import math
for a in range(1,147):
    for b in range(a+1,147):
        if 4*a*b == (147-a-b)*(147-a-b) and a + b < 147:
            print((a,b))
```

Running this program gives us the answers of $(3, 108)$, $(12, 75)$ and $(27, 48)$.