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# Grade 6 Math Circles

Wednesday, February 17, 2021

***The Golden Ratio***

## Introduction

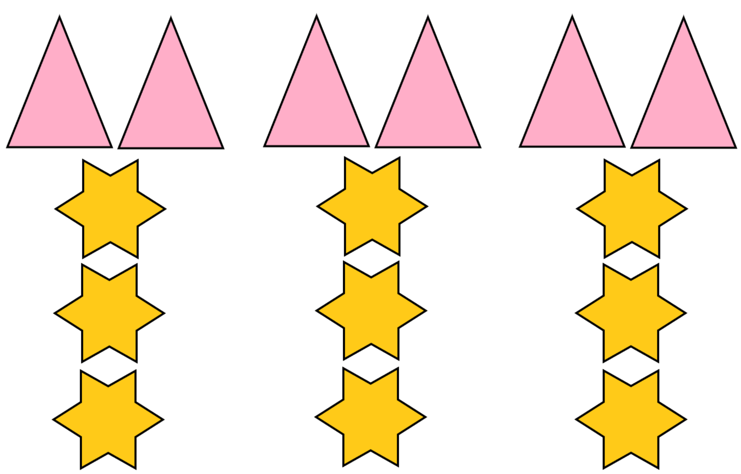
When you think of mathematics, what comes to mind? Some of the first things that pop up are probably numbers, equations, shapes, and graphs. However, there is an artistic aspect to mathematics. This week, we are going to look at one mathematical dimension of what makes something aesthetically pleasing - **the Golden Ratio**.

## Ratios and Proportions

Before we discuss a special ratio like the Golden Ratio, let's become familiar with the ideas behind ratios.

A **ratio** is a comparison of two things. It describes how much of one object there is in relation to another. For example, how many teachers there are to students in a classroom, apples to oranges at a fruit stand or markers to crayons in a pencil case. We mathematically write ratios using a ':' to separate both parts and represent the quantity of each item using whole numbers.

(The number of some item):(The number of some other item)



**Example 1:** The picture above shows 6 triangles and 9 stars. So, we can say there are 6 triangles to 9 stars or the ratio of triangles to stars is 6 : 9. A fraction of $\frac{6}{9}$ can also be used to represent the ratio.

Sometimes, we have large numbers in a ratio that can be simplified.

**Example 1 (continued):** If we look at the ratio of triangles to stars as a fraction, we notice that the fraction $\frac{6}{9}$ can be simplified. Both 6 and 9 are divisible by 3 so,

$$\frac{6}{9} \xrightarrow{\div 3} \frac{2}{3}$$

Likewise, we can simplify the ratio 6 : 9.

$$6 : 9 \xrightarrow{\div 3} 2 : 3$$

**Question:** Does the fraction $\frac{6}{9}$ represent how many shapes are triangles? If not, what does it represent?

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**Answer:** While $\frac{6}{9}$ is another representation of the ratio of triangles to stars, it does not mean that $\frac{6}{9}$ of the shapes are triangles. Since we have 6 stars and 9 triangles, we have $6 + 9 = 15$ shapes in total. The fraction to represent how many of the 15 shapes are triangles would then be $\frac{6}{15}$ or simply $\frac{2}{5}$.

Simplifying a ratio by dividing does not change its value. The ratio remains the same if we do the opposite and multiply it to get a larger ratio, as long as we multiply both parts of the ratio by the same number.

**Example 2:** A brownie recipe uses 2 eggs to 1 cup of flour. Therefore, the ratio of eggs to cups of flour is 2 : 1. Benji bought a dozen eggs to make brownies using this recipe. How many cups of flour will he need?

**Solution:** Benji has 6 times the original quantity of eggs required so we can multiply both parts of our ratio by 6 to figure out how many cups of flour he will need.

$$2 : 1 \xrightarrow{\times 6} 12 : 6$$

So, Benji will need 6 cups of flour if he uses a dozen eggs.

Ratios help us solve a number of problems since they can be represented in numerous ways. **Rates** are very similar to ratios except they specifically have a time element. For example, if Rayyan can run at a rate of 7 m/s, we can represent this as a ratio of 7 meters : 1 second. To solve for a rate, we can use the same strategy that we use for ratios: first find the rate for a single unit and then multiply by the number of units we want.

**Example 3:** As a summer job, Jeremy paints walls. If he can paint 2 walls every hour and charges \$15 per hour, how many walls will he have to paint to earn \$165?

**Solution:** Jeremy charges 15/hour which can be represented as 15 dollars : 1 hour. Note that $165 \div 15 = 11$ or

$$15 : 1 \xrightarrow{\times 11} 165 : 11$$

So, it will take 11 hours for Jeremy to earn \$165. The rate at which Jeremy paints is 2 walls/hour which can be represented as 2 walls : 1 hour. Using equivalent ratios we see that

$$2 : 1 \xrightarrow{\times 11} 22 : 11$$

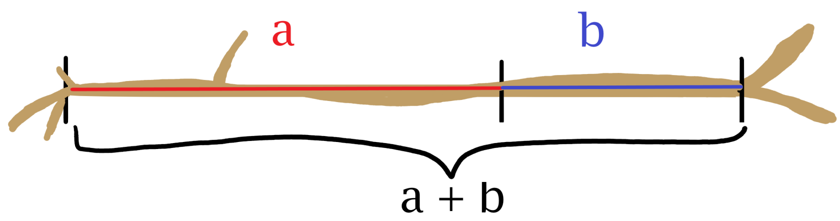
Therefore, Jeremy has to paint 22 walls to earn \$165.

## Phi - The Golden Ratio

The **golden ratio** or the number phi, $\phi$, is a mathematical concept that dates back to Ancient Greece. It is considered to represent perfect beauty or be found uniquely in nature but such claims remain to be proven.

In this section, we are going to solve for the value of $\phi$ and its relation to other concepts in math.

Imagine you have a stick that you can break into two parts. If the ratio between the two portions is the same as the ratio between the whole stick and the larger segment, it is said that the pieces are in golden ratio.




Basically, we find the golden ratio when we divide a line into two parts such that, the long part divided by the short part *equals* the whole length divided by the long part.

$$\frac{a}{b} = \frac{a+b}{a} = \phi$$

## Formula

Let's come up with a formula for phi. Since the Golden Ratio has this property: $\frac{a}{b} = \frac{a+b}{a}$, we start by splitting up the right-hand fraction.

Recall that when adding fractions with the same denominator, we simply add the numerators. Here we are reversing that process to produce two fractions. So,

$$\frac{a}{b} = \frac{a+b}{a}$$
$$\frac{a}{b} = \frac{a}{a} + \frac{b}{a}$$

The Golden Ratio, $\phi$, is $\frac{a}{b}$ while $\frac{a}{a} = 1$ and $\frac{b}{a} = \frac{1}{\phi}$.

*Explore:* Watch the following video to see how we came to $\frac{b}{a} = \frac{1}{\phi}$ using fractions and reciprocals: https://youtu.be/hhzoCrbblDw.

Therefore, the formula we get is

$$\phi = 1 + \frac{1}{\phi}$$

Phi can be defined in terms of itself! We can now use the formula to try and calculate the Golden Ratio.

## Calculation

Fill out the following table with the aid of a calculator. Click here to fill the table online.

First *guess* a positive value for $\phi$. Then, repeat the following calculation for each row of the table.

1. Divide 1 by your value.
2. Add 1.
3. Now use *that* value and start again from step 1.

**Example 4:** Suppose our first guess was 1.5. Then, the first rows of our table would look like

| value | $\frac{1}{\text{value}}$ | $\frac{1}{\text{value}} + 1$ |
|---|---|---|
| 1.5 | $\frac{1}{1.5} = \frac{2}{3} = 0.666...$ | $1 + 0.666... = 1.666...$ |
| 1.666... | $\frac{1}{1.666...} = \frac{1}{\frac{5}{3}} = 0.6$ | $0.6 + 1 = 1.6$ |

A full video solution to this example can be found at the following link: `https://youtu.be/LopTE6hjbco`

*Note: Try to be as accurate as possible and use as many decimal places as your calculator allows. If you do need to round, round to at least 3 decimal places.*

| value | $\frac{1}{\text{value}}$ | $1 + \frac{1}{\text{value}}$ |
|---|---|---|
| | | |
| | | |
| | | |
| | | |

Phi is an **irrational number**, that is, it has non-repeating digits after the decimal point that keep going on and on. We know that the value of the Golden Ratio, or $\phi$, is:

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803398874989484...$$

## Fibonacci Sequence

The **Fibonacci Sequence** is a special series of numbers that is connected to the Golden Ratio. Before we get to their relationship, let's define the Fibonacci Sequence.

The **Fibonacci Sequence** is a list of numbers that starts with $0, 1$. The next number in the sequence is found by adding the two numbers before it. Since $0 + 1 = 1$, the next number in the sequence is 1. Next, since $1 + 1 = 2$, the following term is 2. Here are the first 10 terms of the Fibonacci Sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34$$

*Exercise:* Calculate the next 5 terms of the Fibonacci Sequence. Watch the solution using this link: https://youtu.be/KLxb_L7qJS0.

Now, if you take any two consecutive Fibonacci Numbers, their ratio is very close to the Golden Ratio. In fact, the larger the pair of Fibonacci Numbers, the closer the approximation.

| $a$ | $b$ | $\frac{b}{a}$ |
|---|---|---|
| 2 | 3 | 1.5 |
| 3 | 5 | $1.\overline{66}$ |
| 5 | 8 | 1.6 |
| 8 | 13 | 1.625 |
| ... | ... | ... |
| 144 | 233 | 1.618055556... |
| 233 | 377 | 1.61802575 |

*Exercise:* Find two Fibonacci numbers that have a ratio of approximately 1.618034. The solution can be found at: https://youtu.be/kV0faGqIztI.