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# Grade 6 Math Circles

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***Set Theory***

## Introduction to Sets

A **set** is a collection of unique objects. The objects in a set are called **elements** of the set.

Set notation is fairly simple. List each element of the set between a pair of curly brackets with each element separated by a comma. The curly brackets, '{ }', are sometimes called **set brackets**, or braces.

**Example 1:** Use set notation to describe the different sets.

a) colours of the rainbow

b) Canadian provinces

c) children's TV shows

**Solution:**

a) Using the acronym 'ROYGBIV', we get {red, orange, yellow, green, blue, indigo, violet}.

b) Listing the provinces from West to East, we get {British Columbia, Alberta, Saskatchewan, Manitoba, Ontario, Quebec, Newfoundland and Labrador, New Brunswick, Nova Scotia, Prince Edward Island}.

c) {Dora the Explorer, Barney and Friends, Paw Patrol, The Wiggles, Blue's Clues, Arthur,...}

*Note: The order of the elements in the set does not necessarily matter. Therefore, the set {blue, green, indigo, orange, red, violet, yellow} is alternative solution to part a).*

For part (c), we end the set with "..." or an *ellipsis*. An ellipsis represents continuation and that means that the set of children's TV shows continues on...forever.

*Note: The list of children's TV shows does not really go on forever. However, since new shows are continuously being released around the world, we are going to say it is infinite for this example.*

As such, we call the sets in part (a) and (b), **finite sets**, while the set in part (c) is called an **infinite set**.

*Exercise 1:* Write the set of days of the week using set notation. Is this a finite or infinite set?

We can also use an ellipsis in the middle when dealing with long lists. In this case, we are omitting elements in the middle of the set when writing the set.

**Example 2:** Consider the set of letters in the alphabet. We can write it as {a, b, c, ..., x, y, z}. This is still a finite set since there are only 26 letters in the alphabet but we use an ellipsis in the middle to be more efficient.

## More Set Notation

It is common for sets to be represented using a capital letter. For example, $A = \{1, 2, 3, 4, 5\}$. To show that an element is within a set, we use the $\in$ symbol meaning "is an element of". So, we can say that $2 \in A$. Likewise, the $\notin$ symbol denotes that an object "is not an element of" a set. In our example, $0 \notin A$.

**Example 3:** Let $X$ be the set of the first 50 positive multiples of 2. Determine which of the following statements are true or false.

a) $23 \in X$ b) $106 \notin X$ c) $88 \in X$

**Solution:**

a False. The number 23 is not even and not a multiple of 2. Therefore, it cannot be an element of $X$.

b True. Note that 106 is not one of the first 50 multiples of 2.

c True. The 44th multiple of 2 is 88.

A set with no elements, $\{\}$, is called the **empty set** and denoted using the $\emptyset$ symbol.

## Cardinality

The **cardinality** of a set is the size of the set. We can determine the cardinality of a finite set by determining the number of elements in the set. On the other hand, an infinite set has infinite cardinality.

**Example 4:** The set $A = \{100, 101, 102, 103\}$ has 4 elements. We can write the cardinality of the set $A$ as $|A| = 4$.

Recall from our definition of sets that each element in a set is unique. A set may have duplicate objects, however, we omit these duplicates before taking the cardinality of the set.

**Example 5:** The set $A = \{100, 101, 101, 100, 104, 104, 106, 104, 104, 106\}$ has 10 elements but all of these elements occur multiple times in the set! When we take out the repetitions, we get $A = \{100, 101, 104, 106\}$ so, we can write the cardinality of the set $A$ as $|A| = 4$.

## Subsets

A collection may not contain all the elements of a set. For example, Ruobing owns three Harry Potter books. While her collection is a set, it is also a **subset** of the whole Harry Potter series.

Consider a set $B$.

A set $A$ is a **subset** of $B$ if and only if every element of $A$ is in $B$. We write this as $A \subseteq B$. The symbol $\nsubseteq$ shows that a set is not a subset of another.

A set $A$ is a **proper subset** of $B$ if $A$ is a subset of $B$ such that, at least <u>one</u> element in $B$ is not in $A$. That is, $A$ is a subset that is not identical to the original set and contains fewer elements. When $A$ is a proper subset of $B$, we write this as $A \subset B$. We use the symbol $\not\subset$, when a set is not a proper subset of another.

Let's use these definitions in an example.

**Example 6:** Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{1, 3, 5, 7\}$. The set $B$ is a subset of $A$, or $B \subseteq A$ since all the elements of $B$ are in $A$. Furthermore, $B$ is a proper subset of $A$ since they are not equal which can be stated as $B \subset A$. If we had $C = \{3, 5, 1, 7\}$, then technically $C$ is a subset of $B$. However, we would not call it a proper subset of $B$.

*Exercise 2:* Define a set that is not a subset of $A$.

Two sets are considered **equal** if they are subsets of each other i.e. they have the same elements. Since order does not matter, the sets $B$ and $C$ from the example above are equal and we write this as $B = C$.

*Explore:* The Universal Subset

The empty set, $\emptyset$, is a subset of all sets. Here's a short explanation:

Since it contains no elements, every element of the empty set belongs to any set. By definition, the empty set is a subset of all sets.

Note that the empty set is the smallest set in existence. As such, it will always have fewer elements and not be identical to any set that is not the empty set. Therefore, the empty set is a proper subset of all sets with at least one element.

## Set Operators

Often, sets interact with one another. For example, two sports teams have their own set of players that come together to compete at a game.

Consider two sets, $A$ and $B$.

The **union** of two sets contains the combined elements from both sets. So, an element of either $A$ OR $B$ will be in the union of both sets, $A \cup B$.

The **intersection** of two sets contains the elements that are present in both sets. So, the intersection of both sets, $A \cap B$, contains the elements in both $A$ AND $B$.

In the sports teams example, the set of players in a game between two teams is the union of the sets of players in each individual team. A player that has played for both teams would be part of the intersection of the two sets of teams.

**Example 7:** Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{1, 3, 5, 7\}$. The union of the two sets is $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$. Note we do not include duplicates of the numbers $1, 3, 5, 7$.

The intersection of the two sets $A$ and $B$ are the elements that are in both sets. So, $A \cap B = \{1, 3, 5, 7\}$.

We can generalize this idea to compute the unions and intersections of more than two sets.

**Example 8:** Let $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{5, 7, 9, 11, 13\}$ and $C = \{0, 5, 10, 15\}$. The union of the three sets is $A \cup B \cup C = \{0, 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 15\}$. Again, we do not include duplicates of the numbers $5, 7$.

Alternatively, we could compute the union of the three sets by first, taking the union of $A$ and $B$. Let $X = A \cup B$ to get $X = \{1, 2, 3, 4, 5, 6, 7, 9, 11, 13\}$. Now, take the union of $X$ and $C$. Therefore, $X \cup C = \{0, 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 15\}$.

The intersection of the three sets $A$, $B$ and $C$ are the elements that are in all three sets. The only element that is in all three sets is 5 so, $A \cap B \cap C = \{5\}$.

*Try this:* Click the link and use the Geogebra app to practice using set operators: https://www.geogebra.org/m/jpdeey7t.

# Universal Set

A **universal set** is a set that contains all the elements required by a set theory problem.

When a universal set is defined, we can look at all the elements that are not in a particular set. This is called the **complement** of a set; a set which contains all the elements in the universal set that are not in the original set. We denote that a set, $A$, is the complement of another set, $B$, by saying $A = B^C$.

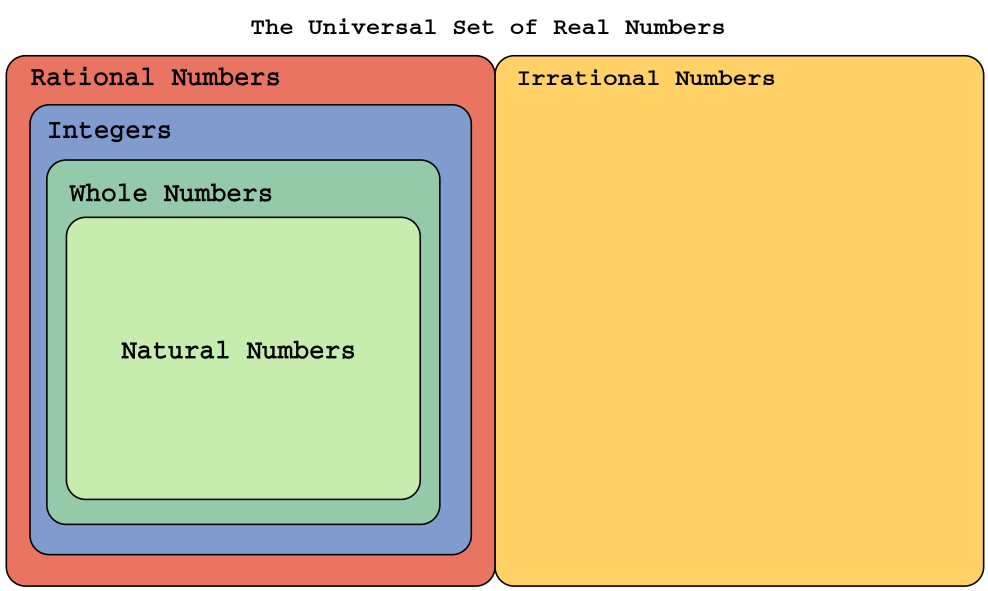
If we were looking at a universal set of students, the teacher may look at the complement of a set of students who are in attendance to contact about missing class.

# Number Systems

A **number system** is a set of numbers. Here are some familiar number systems:

- *Natural Numbers* are the positive whole numbers (i.e. 1, 2, 3, 4,...). There are infinitely many natural numbers.
- *Whole Numbers* include all natural numbers with the addition of 0 (i.e. 0, 1, 2, 3, 4,...).
- *Integers* are whole numbers and their additive inverses - their "opposites" (i.e. ..., -3, -2, -1, 0, 1, 2,...).
- *Rational Numbers* are numbers that can be expressed as a ratio between two integers. Each integer can be expressed as a ratio between itself and 1 (e.g. $4 = \frac{4}{1}$). Rational numbers also include terminating decimals (e.g. $\frac{1}{8} = 0.125$) and non-terminating, repeating decimals (e.g. $\frac{1}{6} = 0.1\overline{66}$).
- *Irrational Numbers* are the numbers that *cannot* be expressed as a ratio between two integers. The decimal form of these do not terminate and have no repetition or pattern. Some famous irrational numbers include $\phi$ (the Golden Ratio), $\pi$ (the ratio between the circumference of a circle to its diameter), and $e$ (Euler's constant). Irrational numbers also include radicals (e.g. $\sqrt{2} = 1.414213562...$).
- *Real Numbers:* the set of numbers that includes "all the numbers" on a number line. More specifically, it includes all rational and irrational numbers.

Take a look at the following figure and observe how these number systems interact.



In math, the universal set is often the set of real numbers although some topics in math use different number systems such as the set of **complex numbers**. As the diagram above illustrates, all real numbers can be categorized into smaller number systems which are subsets of the set of real numbers. Note that within the universal set of real numbers, we can take the complement of the set of rational numbers to be the set of irrational numbers and vice versa.