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CENTRE FOR EDUCATION IN
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Grade 6 Math Circles

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Exponents - Solutions

Exercise: Calculate 2×5 and 2^5. Are the answers the same? Is it possible for $2 \times n$ to equal 2^n where n is an integer? If so, what is a possible n?

Since $2 \times 5 = 10$ and $2^5 = 32$, there are clearly two different answers. However, 2 multiplied by 1 and 2 to the exponent 1 have the same value. This is the case for 2×2 and 2^2 as well.

1. Evaluate.

   (a) $5^3 = 125$
   (b) $\left(\frac{101}{4}\right)^0 = 1$
   (c) $2^6 = 64$
   (d) $1^{27} = 1$
   (e) $4^4 = 256$
   (f) $0^{56} = 0$
   (g) $13^2 = 169$
   (h) $\phi^1 = \phi$

2. Write the following as powers. *(Hint: For part (c), try factoring the number)*

   (a) $333 \times 333 \times 333 = 333^3$
   (b) 91 to exponent 5 $= 91^5$
   (c) $216 = 6^3$
   (d) 5 to the fourth power $= 5^4$
   (e) 100 squared $= 100^2$
   (f) $7 \times 11 \times 11 \times 7 \times 11 \times 7 \times 7 = 7^4 \times 11^3$

3. Use the given base to write the following numbers as powers.

   (a) 4096, base $= 4$; $4096 = 4^6$
   (b) 625, base $= 5$; $625 = 5^4$
   (c) 1000, base $= 10$; $1000 = 10^3$
   (d) 512, base $= 2$; $512 = 2^9$
   (e) 289, base $= 17$; $289 = 17^2$
   (f) 81, base $= 3$; $81 = 3^4$

4. Express 1 as a power that has a base not equal to 1.

   Recall that any base, not including 0, raised to the exponent 0 has value 1. So, one possible answer is 123^0.

5. Evaluate.

(a) $(-30)^4 = 30^4 = 810,000$

(b) $47^{-1} = \left(\frac{1}{47}\right)^1 = \frac{1}{47}$

(c) $2^{-6} = \left(\frac{1}{2}\right)^6 = \frac{1^6}{2^6} = \frac{1}{64}$

(d) $(-1)^{38} = 1^{38} = 1$

(e) $(15)^{-3} = \left(\frac{1}{15}\right)^3 = \frac{1^3}{15^3} = \frac{1}{3375}$

(f) $\left(-\frac{7}{6}\right)^5 = \left(-\frac{7^5}{6^5}\right) = \left(-\frac{16,807}{7,776}\right)$

(g) $(-88)^0 = 1$

(h) $\left(-\frac{4}{25}\right)^{-2} = \left(-\frac{25}{4}\right)^2 = \left(\frac{25}{4}\right)^2 = \frac{25^2}{4^2} = \frac{625}{16}$

6. Fill in the table summarizing different properties of exponents.

Property	Explanation	Example
Base of 1	A power with the base 1 will equal 1 for any exponent.	$1^{23456789} = 1$
Base of 0	A power with the base 0 will equal 0 for all exponents except for 0.	$0^{1000} = 0$
Exponent of 1	Any base to the exponent 1 will simply return the value of the base itself.	$\pi^1 = \pi$
Exponent of 0	Any base that is not equal to 0 will equal 1 when raised to the exponent 0.	$\left(-\frac{359^2 + e}{88}\right)^0 = 1$

Negative Exponent	When a power has a negative exponent, we raise the reciprocal of the base to the corresponding positive exponent.	$10^{-5} = \left(\frac{1}{10}\right)^5 = \frac{1}{100,000}$ $\left(\frac{14}{11}\right)^{-2} = \left(\frac{11}{14}\right)^2 = \frac{121}{196}$
Negative Base	A negative base to an odd exponent will result in a negative answer while a negative base to an even exponent will result in a positive answer.	$(-7)^3 = (-1) \times (7)^3 = (-1) \times 343 = -343$ $\left(-\frac{3}{4}\right)^6 = \left(\frac{3}{4}\right)^6 = \frac{729}{4096}$

7. Consider the fraction $\frac{16}{81}$.

   (a) Write $\frac{16}{81}$ as a power with a negative base.

   Note that $\frac{16}{81}$ is $\left(\frac{2}{3}\right)^4$. Since 4 is an even exponent, the power $\left(-\frac{2}{3}\right)^4$ can also be simplified to $\frac{16}{81}$. Therefore, $\frac{16}{81}$ can be written as the power $\left(-\frac{2}{3}\right)^4$

   (b) Write the fraction as a power with a negative base and exponent.

   From part (a), we know that the fraction can be written as the power $\left(-\frac{2}{3}\right)^4$. If we make the exponent negative, we need to take the reciprocal of the base to retain the same value. So, the fraction can be written $\left(-\frac{3}{2}\right)^{-4}$

8. Use exponent rules to simplify the following expressions. Note that part (g)-(h) are challenge questions.

(a) $\left(\frac{12}{7}\right)^2 = \frac{12^2}{7^2} = \frac{144}{49}$

(b) $10^3 \times 10^7 = 10^{3+7} = 10^{10}$

(c) $\frac{6^{38}}{6^6} = 6^{38-6} = 6^{32}$

(d) $(13^{11})^2 = 13^{11\times 2} = 13^{22}$

(e) $\frac{4^{84}}{4^{83}} = 4^{84-83} = 4^1 = 4$

(f) $(9^7)^7 = 9^{7\times 7} = 9^{49}$

(g) $ab^2 \times bc^2 \times abc = a^1b^2 \times b^1c^2 \times a^1b^1c^1 = a^{1+1}b^{2+1+1}c^{2+1} = a^2b^4c^3$

(h) $(8u^5v^4)^{12} = 8^{1\times 12}u^{5\times 12}v^{4\times 12} = 8^{12}u^{60}v^{48}$

9. Fill in the blanks with $<$, $>$, or $=$ to complete the inequality.

(a) $2^4 = 4^2$

(b) $6^{-3} < 6^3$

(c) $99^0 > 0^{45}$

(d) $(-1)^{27} < 1^{27}$

(e) $7^{-3} < 7^{-1}$

(f) $4^{-6} > (-4)^{-7}$

10. The human brain has $\frac{(10^{15})(10^{31})}{(10^{2^3})^{2^2}}$ neuronal connections. Express this number as a single power.

Using exponent rules, we can simplify the expression as follow:

$$\begin{aligned}\frac{(10^{15})(10^{31})}{(10^{2^3})^{2^2}} &= \frac{10^{15+31}}{(10^8)^4} \\ &= \frac{10^{46}}{10^{8\times 4}} \\ &= \frac{10^{46}}{10^{32}} \\ &= 10^{46-32} \\ &= 10^{14}\end{aligned}$$

Therefore, there are 10^{14} neuronal connections in the human brain.

11. According to the Big Bang Model, the universe has $\left(\frac{(5^{26})(5^9)}{5^{34}}\right)^{1^{450}} \times (10^{11})^2$ stars. Simplify the expression.

Using exponent rules, we can simplify the expression as follow:

$$\begin{aligned}
\left(\frac{(5^{26})(5^{9})}{5^{34}}\right)^{1^{450}} \times (10^{11})^2 &= \left(\frac{5^{26+9}}{5^{34}}\right)^{1} \times (10^{11\times 2}) \\
&= \left(\frac{5^{35}}{5^{34}}\right)^{1} \times (10^{22}) \\
&= (5^{35-34})^1 \times 10^{22} \\
&= (5^1)^1 \times 10^{22} \\
&= 5^{1\times 1} \times 10^{22} \\
&= 5^1 \times 10^{22} \\
&= 5 \times 10^{22}
\end{aligned}$$

Therefore, there are 5×10^{22} stars in the universe.

12. Tom had three hamsters at the beginning of the year. After 4 weeks, he now has 243 hamsters. Assuming weekly exponential growth, how much is the population exponentially increasing by every week?

    Assuming the population grows exponentially every week, let x represent the weekly exponential growth. After 4 weeks, the population growth can then be represented by x^4 multiplied by our initial population of 3 hamsters. Note that $3^4 = 81$ and $81 \times 3 = 243$. So, the population is tripling every week.

13. A bacteria population quadruples in size every 10 minutes. If the initial bacteria population is 7 specimens, what is the population after 1 hour?

    Since there are 60 minutes in an hour, the bacteria population will quadruple in size 6 times. The population quadrupling in size 6 times can be represented by the power 4^6 or 4096. Then, multiplying by the initial population, we get that the bacteria population after 1 hour will be $28,672$ specimens.

14. Rio created a computer algorithm which can process an exponential amount of data and reduce processing time. For example, let x represent the initial amount of data the algorithm can process in one minute. Then, it can process x^2 amount of data in 2 minutes. Rio calculates that the algorithm is able to process 400 GB of data in 2 minutes. How long would it take to process $3,200,000$ GB of data?

    Since $x^2 = 400$, we can observe that x is 20. So, the algorithm can process an initial amount of 20 GB of data in 1 minute. Then, since $20^5 = 3,200,000$, we can conclude that it would take the algorithm five minutes to process $3,200,000$ GB of data.

15. Magic Morning Coffee shares that 15% of the caffeine in their drink is consumed by the human body every hour. If a teacher had 1 cup of coffee, or 128 mg of caffeine, at 10 AM this morning, how much caffeine is left in their system at 5 PM?

    If 15% of the caffeine is consumed every hour, then 85% remains. Since 7 hours have passed from when the teacher drank their coffee and there was 128 mg of caffeine to start, we can multiply 128 by the power 0.85^7. This power represents the amount of caffeine left in their system at the end of 7 hours. Therefore, the teacher has approximately 41.034 mg of caffeine in their body at 5 PM.

16. The **half-life** of a substance is the amount of time it takes for the quantity of the substance to decrease to half its original value. For example, gallium-67 is a substance used in nuclear medicine with a half life of 80 hours. If a scientist has 288 grams of gallium-67 at the beginning of an experiment, and 72 grams at the end, how long was the experiment?

    Half of the amount of gallium-67 at the beginning of the experiment is 144 g. Then, half of this amount is 72 g. Thus, the substance has decreased to half its original value twice. In other words, it has gone through 2 half-life periods. Since each half-life is 80 hours, the experiment is 80×2 or 160 hours.