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Grade 6 Math Circles

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Graphing Functions

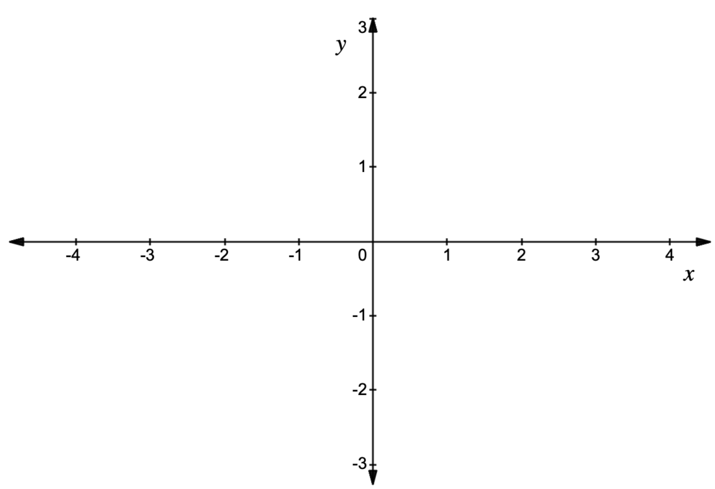
The Cartesian Plane

Let's first look at how a number line works. Along the line, different points represent different numbers. As you move right along the number line, the numbers increase, and as you move left, the numbers decrease.

A different way to think of it is that different numbers represent different positions along this line! You can go farther to the right by increasing the number, and go farther to the left by decreasing the number. We can think of these numbers as **coordinates** for the position that they represent. Let's plot the positions represented by the coordinates (-2) and (5).

This method of being able to translate between numerical coordinates and physical/visual positions is what makes graphing such a powerful tool. Let's also give this number line a different name: an **axis**.

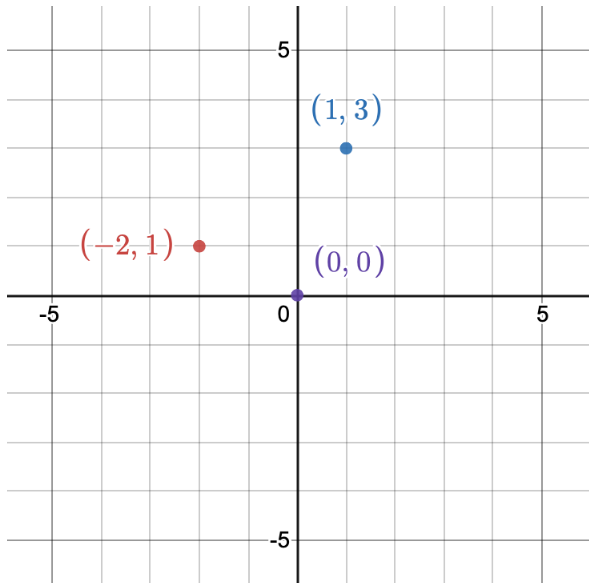
The Cartesian plane has two axes: the horizontal x**-axis** and the vertical y**-axis**. It works as if we put two number lines together: one running left-right, and the other running down-up.



The horizontal x-axis represents the value of x, and as x increases, you move right along the axis. The vertical y-axis represents the value of y, and as y increases, you move upward along the axis. The point where the two axes meet is where both x and y are equal to zero.

We plot points on the Cartesian Plane using coordinates that represent those values of x and y. We write coordinates for the Cartesian Plane like this: (x, y). The horizontal x-coordinate always goes first, and the vertical y-coordinate second. Just like on a number line, the x-coordinate is a number that represents a position along the x-axis, and the y-coordinate represents a position along the y-axis.

Example 1: The coordinates $(0, 0)$, $(1, 3)$, and $(-2, 1)$ are plotted below:



Activity: Use the linked Geogebra apps to practice plotting coordinates:

- $(3, -1)$: `https://www.geogebra.org/m/p6tanwy4`
- $(0, 3)$: `https://www.geogebra.org/m/qjhxsm43`
- $(-2, -2)$: `https://www.geogebra.org/m/fpfw3huv`
- $(-1, 0)$: `https://www.geogebra.org/m/fzmm6vx6`

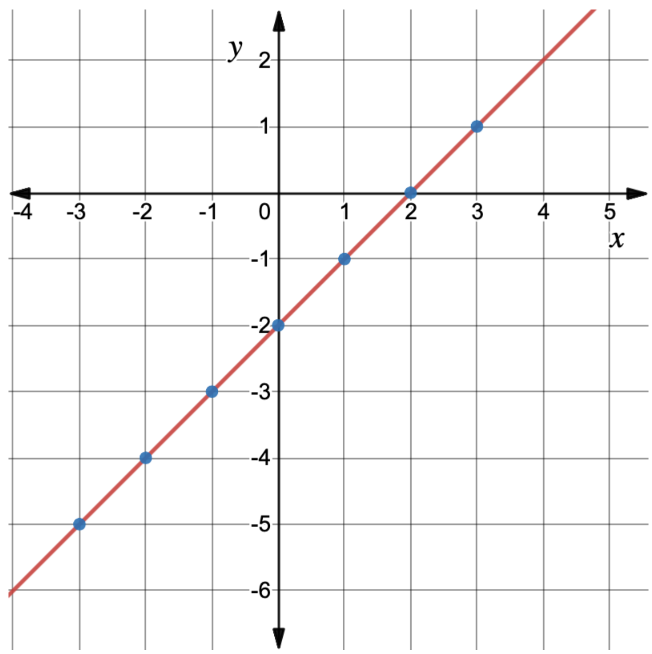
Graphing a Function

To graph a function, we plot the input along the x-axis and the corresponding output along the y-axis. In general, we simply let $y = f(x)$ and treat y as a variable representing the function's output.

Example 2: Let $f(x) = x - 2$.

x	-3	-2	-1	0	1	2	3
$y = x - 2$	-5	-4	-3	-2	-1	0	1

By evaluating values of y for different values of x, we can identify coordinates that are part of the graph of $f(x)$. From this example, we have $(-3,-5)$, $(-2,-4)$, $(-1,-3)$, $(0,-2)$, $(1,-1)$, $(2,0)$, and $(3,1)$. Below, we've plotted these points. We've also "connected the dots" with a straight line. If you imagine plotting the coordinates for every possible input-output pair (x, y), they would altogether form this line!



`https://www.desmos.com/calculator/ddvojrxpxz`

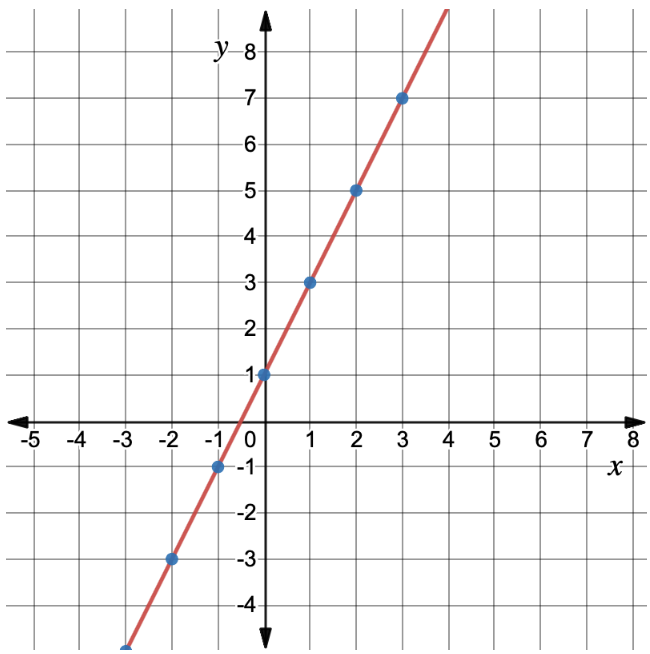
Essentially, a function is graphed by plotting every single coordinate point that represents an input-output pair of the function. Every x value in the domain of a function is represented in the graph. However, much of the time, the domain is very large—even $\mathbb{R}$ has an infinite amount of values in it! Thus, we only calculate (x, y) for a subset of x values from the domain to begin visualizing a function.

Most of the time, the plotted coordinates lay the groundwork for an overall pattern for the function—for instance, we can see that in the example above, the function creates a straight, diagonal line! We can actually check that every point that falls along this line represents an input-output pair of the function $f(x)$. For example, the point $(1.5, -0.5)$ represents the function at $x = 1.5$, where $y = f(1.5) = 1.5 - 2 = -0.5$.

Also, with this function, as x increases, y increases, which is reflected in the upward slope of the line.

Question 1: Let $g(x) = 2x + 1$. Identify coordinates for $x = -3, -2, -1, 0, 1, 2, 3$ in the form (x, y). Plot these points on a graph. Descibe what the function looks like.

Solution: We have $(-3, -5)$, $(-2, -3)$, $(-1, -1)$, $(0, 1)$, $(1, 3)$, $(2, 5)$, and $(3, 7)$. The function produces a straight, diagonal line that is steeper than the previous graph. As x increases, y increases, so the line slopes upward.

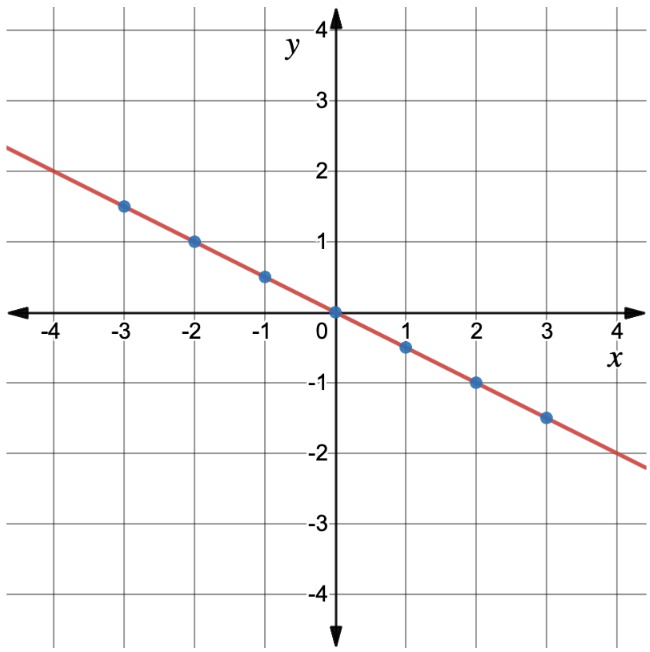


https://www.desmos.com/calculator/l9ufzfoxx1

Notice that for each x-coordinate, there is only ever one y-coordinate—there are never any points stacked "on top" of each other. This reflects an important quality about functions: for each given input, the function can only produce one output. Thus, functions pass what is called the **vertical line test**. To pass the vertical line test, any vertical line that you can draw should intersect the graph at most once.

Question 2: Let $h(x) = -0.5x$. Identify coordinates for $x = -3, -2, -1, 0, 1, 2, 3$ in the form (x, y). Plot these points on a graph. Descibe what the function looks like.

Solution: We have $(-3, 1.5)$, $(-2, 1)$, $(-1, 0.5)$, $(0, 0)$, $(1, -0.5)$, $(2, -1)$, and $(3, -1.5)$. The function produces a straight, diagonal line that is less steep than the previous graphs. As x increases, y decreases, so the line slopes downward.



https://www.desmos.com/calculator/eqixuinjan

The functions that we have graphed so far are called **linear functions** (pronounced "lin-ee-er"), which are functions whose graphs are straight lines. Linear functions take the form $f(x) = mx + b$, where m and b are numbers that change from function to function. For instance, in the function $g(x) = 2x + 1$ from Question 1, $m = 2$ and $b = 1$.

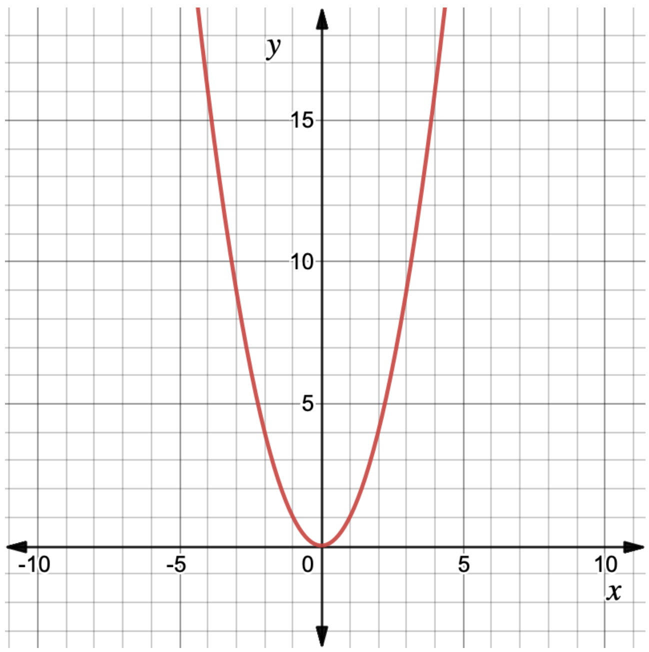
Activity: Linked below is an app where you can control two sliders for the values of m and b. Explore how changing each value changes what the graph looks like. In particular, pay attention to how steep the line is, which way the line slopes, and where the line intersects with the y-axis: https://www.desmos.com/calculator/ygosdaqk5g
This video explores findings from this activity: https://youtu.be/eMkqJbLB_fw

If you zoom in and out of the graph from the activity, you might notice that no matter how far you zoom in, there are no gaps in the line, and no matter how far you zoom out, the line never ends. This is because of the function's domain, $\mathbb{R}$, which encompasses all numbers on the number line and has no least or greatest number!

Non-Linear Functions

Example 3: Let $j(x) = x^2$. We'll calculate some coordinates to graph and connect the points to visualize the function.

x	-4	-3	-2	-1	0	1	2	3	4
$y = x^2$	16	9	4	1	0	1	4	9	16



This curved shape is called a **parabola**, and it is the result of squaring x. You can interact with the graph of $y = x^2$ here: `https://www.desmos.com/calculator/mpv3eokezx`.

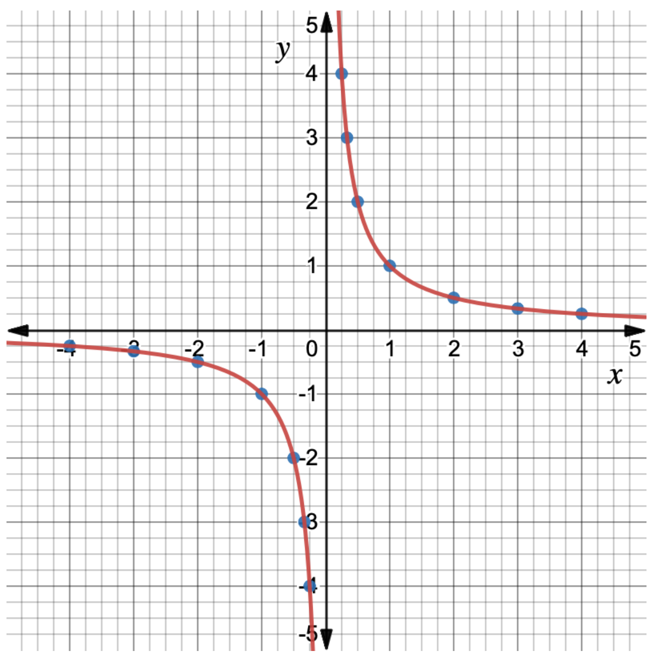
Notice how the y value never goes below zero—this reflects the range of our function! The domain of the function is $\mathbb{R}$, and you will never obtain a negative number by squaring a real number. Thus, the range is $\{y \in \mathbb{R}, y \geq 0\}$, and the y value is never negative!

Another aspect to notice is that the two "arms" of the parabola are symmetrical, and they get steeper and steeper as x gets further away from zero. That is to say, as x increases constantly, the value of $y = j(x)$ increases faster and faster. Why is this? (Think about how exponents work!)

Rational Functions & Asymptotes

Example 4: Let $k(x) = \frac{1}{x}$. We'll calculate some coordinates and connect the points to visualize the function's graph.

x	-4	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3	4
$y = \frac{1}{x}$	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	-3	-4	undefined	4	3	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

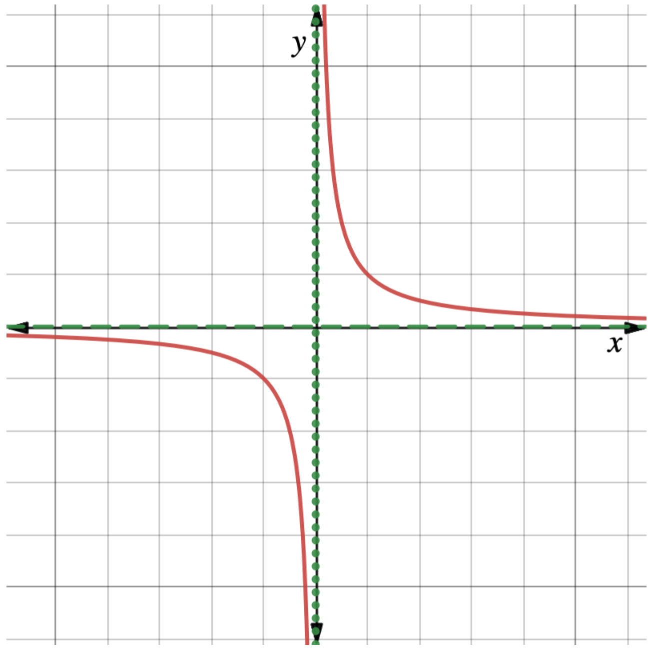


Interact with the graph of $y = \frac{1}{x}$ here: `https://www.desmos.com/calculator/phxb1f6s5l`.

This is a pretty weird-looking graph! The two "halves" are totally separate, and they curve in a way that seems to get closer and closer to the axes but never touch them. In fact, you could draw a vertical line (the dotted green line in the next image) that crosses through $x = 0$, and it would never intersect with the graph, because there's no y value for $x = 0$. That makes sense: $k(x)$ is undefined, after all!

You could do the same thing with a horizontal line (the dashed green line in the next image) that intersects with $y = 0$, because there's no value of x that would give $k(x) = 0$. However, as x gets bigger, $k(x)$ gets smaller, which is why the graph gets closer to this horizontal line, even though they never touch.

These two lines that the graph gets closer to but never touches are called **asymptotes**, and for the graph of $y = \frac{1}{x}$, they look like this:



https://www.desmos.com/calculator/0rcbekirga

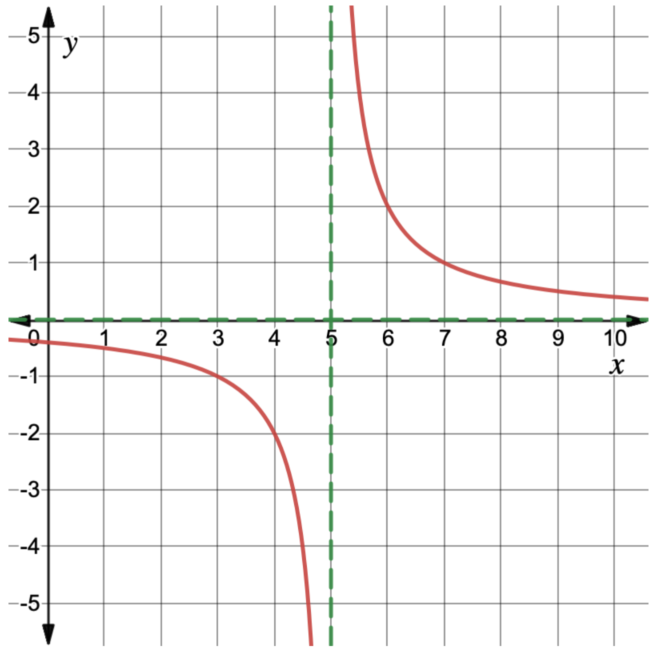
Seeing these asymptotes visually is also helpful for identifying the domain and range! Recall that the domain of $k(x) = \frac{1}{x}$ is $\{x \in \mathbb{R},\ x \neq 0\}$. You can visually see this because of the vertical asymptote at $x = 0$. The horizontal asymptote at $y = 0$, coupled with no indication that any other horizontal asymptotes exist, tells us that the range of the function is $\{y \in \mathbb{R},\ y \neq 0\}$.

Question 3: Graph the function $y = l(x) = \frac{2}{x-5}$. Use the graph to identify the function's asymptotes, domain, and range.

Solution: There is a vertical asymptote at $x = 5$. As x gets closer and closer to 5, the graph of $l(x)$ gets closer and closer to the vertical line at $x = 5$ without ever touching it. This makes sense, since x cannot be equal to 5 because $l(5)$ is undefined.

There is a horizontal asymptote at $y = 0$. As x gets farther and farther from 5, $l(x)$ gets closer and closer to zero. However, x cannot be equal to zero because there is no value of x such that $l(x) = 0$.

Thus, the domain is $\{x \in \mathbb{R},\ x \neq 5\}$, and the range is $\{y \in \mathbb{R},\ y \neq 0\}$.



https://www.desmos.com/calculator/uddmrc5qeu