CEMC Math Circles - Grade 9/10

Wednesday, March 3, 2021
Colouring Fun with Graphs

You Will Need:

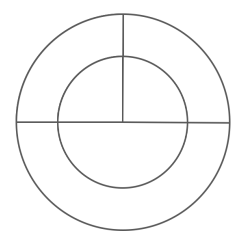
- Pieces of paper
- Pencil crayons or markers for colouring

Colouring Figures

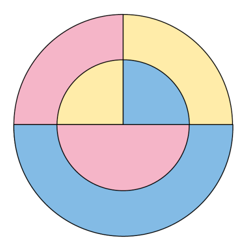
In the following activities, we will be colouring figures using different colours. Here we outline the properties of a *valid colouring* for the purposes of today's activities:

- All regions in the figure are coloured.
- Any pair of neighbouring regions in the figure (that is, regions sharing an edge or border) are coloured with two different colours.
- Regions that meet only at a single point can be coloured with the same colour.

Consider the example shown to the right. Since there are six different regions in the top figure shown, one easy way to produce a valid colouring of the figure is to use six different colours and colour each region using one of these colours. However, the figure can be coloured according to the rules using fewer than six colours. Can you figure out how many colours are actually *needed* to colour this figure?

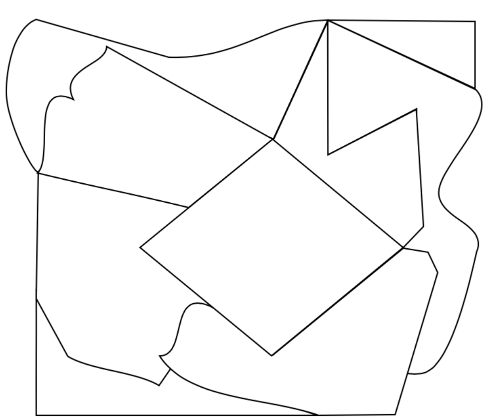


It turns out that there are valid colourings of this figure that only use three different colours, but no valid colouring that uses fewer than three colours. This means that the *minimum* number of colours needed to colour this figure is 3. The bottom image to the right shows one possible valid colouring using three colours. Can you explain why this figure cannot be coloured using only two colours?

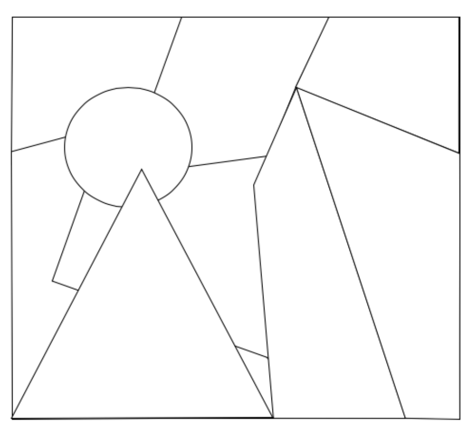


Activity 1: Find a valid colouring for each of the figures shown below that uses the fewest colours possible. Can you explain why the figure cannot be coloured using fewer colours than what you have?

(a)

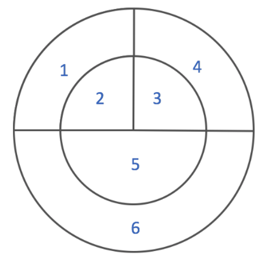


(b)



Modelling Figures Using Graphs

As figures become more complex, it can become difficult to see how to colour them using the fewest colours possible. To help with this, we can translate all of the necessary information needed for colouring from the original figure into a simpler figure called a *graph*. In other words, we *model* the figure using a graph. We can then colour the graph instead of the figure, and then translate this colouring back to the figure. Let's investigate this idea by revisiting the circle figure from earlier.

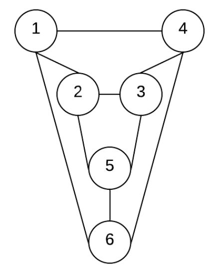


To model a figure using a graph for the purposes of colouring, follow these steps:

i. Label each region of the figure with a unique positive integer.

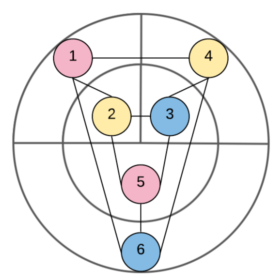
ii. Model the picture from i. using a *graph*. Graphs consist of points (called *vertices*) and lines (called *edges*). Create a point (vertex) in the graph for each region of the picture. We will actually use a circle (as shown) as we would like to include the region number with the point. Next, connect various pairs of vertices using lines (edges). Two vertices should be connected by an edge exactly if they represent neighbouring regions in the original figure.

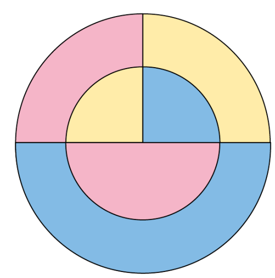
For example, since regions 1 and 2 share a border in the figure, there is an edge between vertices 1 and 2 in the graph, and since regions 4 and 5 do not share a border in the figure, there is no edge between vertices 4 and 5 in the graph.



iii. Colour all of the vertices in the graph so that no adjacent vertices (vertices connected by an edge) share the same colour.

iv. Transfer the colours from the vertices of your graph to the corresponding regions in the original figure. This must correspond to a valid colouring of the figure. (Can you see why?)





Since the original goal was to colour figures using the fewest colours possible, the goal when modelling these problems with graphs is to colour graphs using the fewest colours possible.

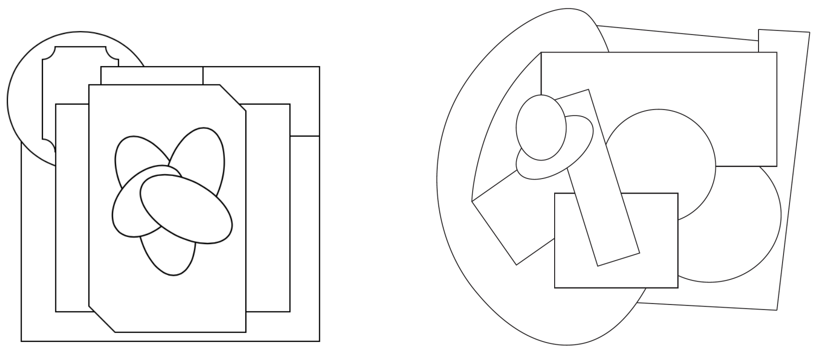
For example, here is the method that was used to colour the graph above: Start with vertex 1, and assign it a colour (pink). Next, identify all of the vertices that are adjacent to this vertex, which in this example are 2, 4 and 6. If possible, colour all of these vertices with the same second colour (yellow). If this is not possible, colour as many as you can with the second colour. In this example, we colour 2 and 4 yellow, but not 6 as it is connected to 4. We now know we need a third colour (blue). It turns out that we can colour all remaining vertices with these three colours.

We can also argue that three colours is the best we can do. Notice that we have no choice but to use three different colours for the three vertices 2, 3, and 5. You need two different colours for 2 and 3, but neither of these colours can then be used for 5.

Activity 2:

(a) Consider the two figures shown below. For each figure, do the following:

- Model the figure using a graph as outlined in i. and ii. on the previous page.
- Find a colouring of the graph as outlined in iii. on the previous page that uses the fewest colours possible.
- Explain why the graph cannot be coloured according to the guidelines using fewer colours than what you have.



If you complete these steps, you will have determined the minimum number of colours needed for a valid colouring of each figure. If you would like, you can now colour the original figures as outlined by your graph colourings!

(b) Can you create a two-dimensional figure that requires five colours in order to achieve a valid colouring? Spend some time thinking about whether or not you think this is possible.

Graph Colouring in Action!

Many real-world problems can be translated into graph colouring problems. These problems often involve resources (colours) and conflicts (two regions that cannot be coloured the same), and you are tasked with assigning resources in an optimal way (using the fewest colours possible) while ensuring that no conflicts arise. This type of problem often arises when attempting to make schedules and timetables, and large scale versions are famously difficult to solve!

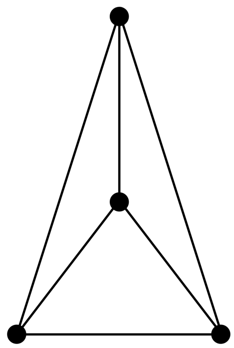
Check out the problems Timetabling and Aircraft Scheduling from past Beaver Computing Challenges. Read the story for the problem and try to figure out how to model this problem as a graph colouring problem. Ask yourself the following questions:

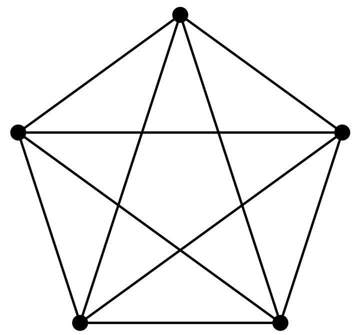
- What should the vertices represent?
- How do you decide whether or not an edge should be drawn between a particular pair of vertices?
- What do the colours represent?
- Does finding a valid colouring of your graph which uses the fewest colours possible provide a solution to the problem?

Planar Graphs

Above, we saw that figures can be modelled by graphs, which in turn makes colouring of figures simpler. The colouring of graphs is related to several problems and theorems in *graph theory*. Graph theory is an area of mathematics devoted to the study of graphs. Along with applications to other areas of mathematics, graph theory has applications in computer science, biology, physics, chemistry, and many more areas of study. There are several different types of graphs, but in this activity, we will take a close look at a special type called *planar graphs*.

The graphs associated to one-dimensional figures, like the ones above, are called *planar graphs*. In fact, every one-dimensional figure has a corresponding planar graph. A planar graph is a graph that can be embedded in the plane, that is to say, it can be drawn on a piece of paper in such a way that no edges cross each other. For some examples, consider the following graphs:

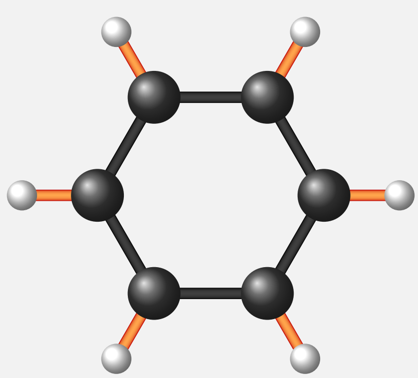


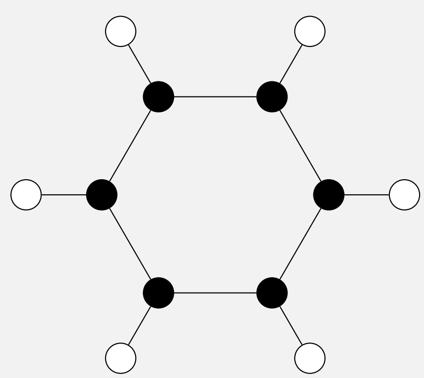


The first graph is a planar graph, while the second graph is not a planar graph. Can you explain why the second graph is not a planar graph?

Graph Theory in Chemistry!

Graphs can be used to model many real-world problems and phenomena. For example, graphs can be used to model molecules in chemistry. Modelling molecules with graphs allows us to gain insight into the physical properties of the molecules. Many physical properties, such as the boiling point of a substance (the temperature at which the substance can change its state from a liquid to a gas), are related to the geometric structure of the molecules in the substance.





The picture on the left is a *benzene molecule*. To the right of the benzene molecule is its corresponding *molecular graph*; the atoms (balls) of the molecule are converted into nodes and the chemical bonds (sticks) of the molecule are converted into edges. The molecular graph of the benzene molecule is an example of a planar graph!

Activity 3:

Consider the planar graph shown below. In graph theory, this planar graph is commonly referred to as the *butterfly graph*. Complete the following tasks:

(a) Find a colouring of the butterfly graph as outlined in iii. on page 2 that uses the fewest colours possible.

(b) In activity 2, we modelled figures using graphs. In this activity, we will do the opposite. Find a one-dimensional figure whose corresponding graph is the butterfly graph. Hint: Can we reverse the process outlined on page 2?

Note: there are a few different figures that you can draw, all of which are valid.

(c) Does your one-dimensional figure look like a butterfly? If not, try to find one that does.

(d) How can you verify that your figure(s) indeed correspond to the butterfly graph?

(e) If you would like, you can colour the figure(s) you found as outlined by your colouring of the butterfly graph!