CEMC Math Circles - Grade 9/10

Wednesday, March 31, 2021
More Counting - Solution

A permutation will be called a **VALROBSAR** permutation if *no* integer in the permutation has two neighbours that both are less than it.

Two integers in the permutation are neighbours if they appear directly beside each other.

Problems:

1. List all permutations of the integers 1, 2, 3, and 4.
2. How many of the permutations of the integers 1, 2, 3, and 4 are VALROBSAR permutations?
3. How many VALROBSAR permutations are there of the integers 1, 2, 3, 4, and 5?
4. How many VALROBSAR permutations are there of the integers 1, 2, 3, 4, 5, and 6?

Solutions:

1. There are 24 permutations of the four integers:

1 2 3 4	1 2 4 3	1 3 2 4	1 3 4 2	1 4 2 3	1 4 3 2
2 1 3 4	2 1 4 3	2 3 1 4	2 3 4 1	2 4 1 3	2 4 3 1
3 1 2 4	3 1 4 2	3 2 1 4	3 2 4 1	3 4 1 2	3 4 2 1
4 1 2 3	4 1 3 2	4 2 1 3	4 2 3 1	4 3 1 2	4 3 2 1

2. There are 8 VALROBSAR permutations. They are the red permutations in the table below:

1 2 3 4	1 2 4 3	1 3 2 4	1 3 4 2	1 4 2 3	1 4 3 2
2 1 3 4	2 1 4 3	2 3 1 4	2 3 4 1	2 4 1 3	2 4 3 1
3 1 2 4	3 1 4 2	3 2 1 4	3 2 4 1	3 4 1 2	3 4 2 1
4 1 2 3	4 1 3 2	4 2 1 3	4 2 3 1	4 3 1 2	4 3 2 1

3. Looking at the solution to 2., you might have noticed that all of the VALROBSAR permutations have the number 4 at one of the two ends. It will be helpful to think about why this must be true. In order to be a VALROBSAR permutation, every number must have at most one neighbour that is smaller than it. Since 4 is the largest among 1, 2, 3, and 4, if a permutation has 4 in one of the two middle positions, then 4 is guaranteed to have two neighbours that are smaller than it. Hence, a permutation cannot be a VALROBSAR permutation unless 4 is at one of the ends. It is worth noting that there are permutations with a 4 at the end that fail to be VALROBSAR permutations, for example, 4132.

   We now focus our attention on the VALROBSAR permutations of 1, 2, 3, 4, and 5. By the same reasoning as in the previous paragraph, a VALROBSAR permutation of these integers must have 5 at one of the ends. The next key observation is that if we "remove" this 5 from the end, what remains will be a VALROBSAR permutation of 1, 2, 3, and 4. This is because removing a number on the end of a permutation does not introduce any new neighbouring pairs, and so cannot cause a failure of the VALROBSAR condition.

   This means all of the VALROBSAR permutations of 1, 2, 3, 4, and 5 take the form $5abcd$ or $abcd5$ where $abcd$ is a VALROBSAR permutation of 1, 2, 3, and 4. On the other hand, if we take a VALROBSAR permutation of 1, 2, 3, and 4 and place a 5 on either end, what results is a VALROBSAR permutation of 1, 2, 3, 4, and 5. To see this, suppose $abcd$ is a VALROBSAR permutation of 1, 2, 3, and 4 and consider the permutation $5abcd$. The neighbours of b, c, and d in $5abcd$ are the same as they are in the permutation $abcd$. Since we are assuming $abcd$ is a VALROBSAR permutation, each of a, b, and c has at most one neighbour smaller than it in $abcd$, and hence, has at most one neighbour smaller than it in $5abcd$. We know that a is equal to one of 1, 2, 3, and 4, so $a < 5$, which means a has at most one neighbour smaller than it in $5abcd$ (namely, b *could* be smaller than a). The number 5 is at the end of the permutation, so it cannot possibly cause a failure of the VALROBSAR condition.

   We are now able to quickly count the number of VALROBSAR permutations of 1, 2, 3, 4, and 5. Using the discussion above, we get all of *these* VALROBSAR permutations by taking a VALROBSAR permutation of 1, 2, 3, and 4 and placing a 5 on one of the two ends. There are 8 VALROBSAR permutations of 1, 2, 3, and 4, so this gives $2 \times 8 = 16$ VALROBSAR permutations of 1, 2, 3, 4, and 5. Furthermore, each of these 16 VALROBSAR permutations must be different. *Can you see why?*

   The VALROBSAR permutations of 1, 2, 3, 4, and 5 are given below:

1 2 3 4 5	2 1 3 4 5	3 1 2 4 5	3 2 1 4 5
4 1 2 3 5	4 2 1 3 5	4 3 1 2 5	4 3 2 1 5
5 1 2 3 4	5 2 1 3 4	5 3 1 2 4	5 3 2 1 4
5 4 1 2 3	5 4 2 1 3	5 4 3 1 2	5 4 3 2 1

   Note: There are more direct ways of counting these permutations without building on the permutations of 1, 2, 3, and 4. (An idea of this form will be discussed in the Extension on the last page.) The method presented above doesn't just give us an easy to count the "next order" of VALROBSAR permutations, but also gives an easy way to list them (based on the "previous list"). When using permutations in mathematics, sometimes we are interested in only the count, and sometimes we are interested in the actual list of permutations. It is often helpful to think about building them in "stages" like we have done here.

4. Similar to the argument in the previous solution, a VALROBSAR permutation of 1, 2, 3, 4, 5, and 6 must have the 6 at one of the ends and what remains after removing the 6 must be a VALROBSAR permutation of 1, 2, 3, 4, and 5. Furthermore, we get a VALROBSAR permutation of 1, 2, 3, 4, 5, and 6 by taking any VALROBSAR permutation of 1, 2, 3, 4, and 5 and adding a 6 to either end of the permutation. For every choice of a VALROBSAR permutation of 1, 2, 3, 4, and 5 and choice of which side to add the 6, we get a VALROBSAR permutation of 1, 2, 3, 4, 5, and 6. There are 16 VALROBSAR permutations of 1, 2, 3, 4, and 5, so this means there are $2 \times 16 = 32$ VALROBSAR permutations of 1, 2, 3, 4, 5, and 6.

Extension: Can you see a pattern forming based on your work in problems 1. to 4.? Suppose that n is a positive integer satisfying $n \geq 2$ and consider the permutations of the integers $1, 2, 3, 4, \ldots, n$. What can you say about the number of VALROBSAR permutations of these integers?

Discussion:

To recap, we found the following counts of the VALROBSAR permutations:

- $n = 4$: $2^3 = 8$ VALROBSAR permutations
- $n = 5$: $2^4 = 16$ VALROBSAR permutations
- $n = 6$: $2^5 = 32$ VALROBSAR permutations

You might guess from this pattern that the number of VALROBSAR permutations of $1, 2, \ldots, n$ is 2^{n-1}, in general. In fact, the arguments in 3. and 4. actually showed why the number of VALROBSAR permutations seemed to double at each stage, and these arguments can be used to justify this formula.

Here is a more direct way to count the number of VALROBSAR permutations of $1, 2, \ldots, n$. Let's build such a permutation by placing each integer in turn and keeping track of how many choices we have at each stage. (This argument could have been used in 3. and 4. as well.)

First, consider n, the largest integer. No matter where you place it in the permutation it will be larger than its neighbours, and so it must have only one neighbour. The integer n must be placed at an end of the permutation, either first or last, which means you have 2 choices.

$$n \_\ \_\ \_\ \ldots\ \_ \qquad\qquad \text{or} \qquad\qquad \_\ \_\ \_\ \ldots\ \_\ n$$

Once you place n, you have $(n-1)$ places left for the remaining integers $1, 2, \ldots, n-1$. As with n above, you have 2 choices for where to place the integer $n-1$: beside n or at the other end of the permutation.

For example, if we placed n as in the left image shown above, then the two leftmost images below show the choices for where to place $n-1$. *Can you see why $n-1$ must be placed like this in order to form a VALROBSAR permutation?* In the other case above, the choices are shown below on the right.

$$n\ (n-1)\ \_\ \_\ \ldots\ \_ \quad \text{or} \quad n\ \_\ \_\ \_\ \ldots\ (n-1) \quad \text{or} \quad (n-1)\ \_\ \_\ \ldots\ \_\ n \quad \text{or} \quad \_\ \_\ \_\ \ldots\ (n-1)\ n$$

After each placement of the largest remaining integer, you then have 2 choices for where to place the next largest integer. This continues until you have placed the integer 2, and then the integer 1 must go in the only remaining place.

Thus, for each of the first $n-1$ integers, $n, (n-1), (n-2), (n-3), \ldots, 2$, you have 2 choices of where each is placed, and then the 1 goes in the last open place. Thus, the number of VALROBSAR permutations of the integers $1, 2, \ldots, n$ is 2^{n-1}.