Intermediate Math Circles
FGH
Problem Set Solutions

Problem 1 Solution

(a) *Solution* (We will use the hint given in the preamble.)

Expressing $\frac{1}{5}$ and $\frac{1}{4}$ with a common denominator of 40, we get $\frac{1}{5} = \frac{8}{40}$ and $\frac{1}{4} = \frac{10}{40}$.
We require that $\frac{n}{40} > \frac{8}{48}$ and $\frac{n}{40} < \frac{10}{40}$, thus $n > 8$ and $n < 10$.
The only integer n that satisfies both of these inequalities is $n = 9$

b) *Solution*

Expressing $\frac{m}{8}$ and $\frac{1}{3}$ with a common denominator of 24, we require $\frac{3m}{24} > \frac{8}{24}$ and so $3m > 8$ or $m > \frac{8}{3}$.
Since $\frac{8}{3} = 2\frac{2}{3}$ and m is an integer, then $m \geq 3$.
Expressing $\frac{m+1}{8}$ and $\frac{2}{3}$ with a common denominator of 24, we require $\frac{3(m+1)}{24} < \frac{16}{24}$ or $3m + 3 < 16$ or $3m < 13$, and so $m > \frac{13}{3}$.
Since $\frac{13}{3} = 4\frac{2}{3}$ and m is an integer, then $m \leq 4$.
The integer values of m which satisfy $m \geq 3$ and $m \leq 4$ are $m = 3$ and $m = 4$.

(c) *Solution*(We will use some of the process we used in (b).)

At the start of the weekend, Fiona has played 30 games and has w wins, so her win ratio is $\frac{w}{30}$.
Fiona's win ratio at the start of the weekend is greater than $0.5 = \frac{1}{2}$, and so $\frac{w}{30} > \frac{1}{2}$.
Since $\frac{1}{2} = \frac{15}{30}$, then we get $\frac{w}{30} < \frac{15}{30}$, and so $w > 15$.
During the weekend Fiona plays five games giving her a total of $30 + 5 = 35$ games played.
Since she wins three of these games, she now has $w + 3$ wins, and so her win ratio is $\frac{w+3}{35}$.
Fiona's win ratio at the end of the weekend is less than $0.7 = \frac{7}{10}$, and so $\frac{w+3}{35} < \frac{7}{10}$.
Rewriting this inequality with a common denominator of 70, we get $\frac{2(w+3)}{35} < \frac{49}{70}$ or $2(w = 3) < 49$ or $2w + 6 < 49$ or $2w < 43$ and so $w < \frac{43}{2}$.
Since $\frac{43}{2} = 21\frac{1}{2}$ and w is an integer, then $w \leq 21$.
The integer values of w which satisfes $w > 15$ and $w \leq 21$ are $w = 16, 17, 18, 19, 20, 21$.

Problem 2 Solution

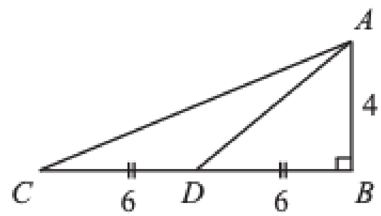
(a) There are three solutions. (The third solution shows a fact that we can use in future questions!!)

Solution 1

In $\triangle ABC, AD$ is a median and so D is the midpoint of BC.
Since $BC = 12$ and D is the midpoint of BC, then $CD = \frac{12}{2} = 6$.
In $\triangle ACD$, base CD has length 6, and corresponding height AB has length 4. (Since $\angle ABC = 90°$, AB is the height of $\triangle ACD$ even though AB is outside $\triangle ACD$.)
Thus, $\triangle ACD$ has area $\frac{1}{2}(6)(4) = 12$.

Solution 2

In $\triangle ABC, AD$ is a median and so D is the midpoint of BC.
Since $BC = 12$ and D is the midpoint of BC, then $CD = DB = 6$.



In $\triangle ABD, AB = 4, DB = 6$, and $\angle ABD = 90°$, and so $\triangle ABD$ has area $\frac{1}{2}(6)(4) = 12$.
Similarly, $\triangle ABC$ has area $\frac{1}{2}(12)(4) = 24$, and so the area of $\triangle ACD$ is the area of $\triangle ABC$ minus the area of $\triangle ABD$, or 24 - 12 = 12.

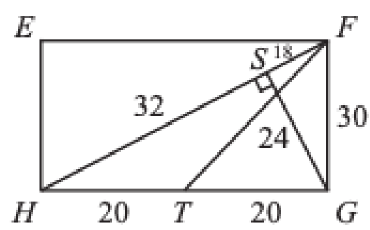
Solution 3 (A median of $\triangle ABC$ divides the the triangle into two equal areas.)

In $\triangle ABC, AB = 4, BC = 12$, and $\triangle ABC = 90°$, so $\triangle ABC$ has area $\frac{1}{2}(12)(4) = 24$.
A median of $\triangle ABC$ divides the the triangle into two equal areas.
Let's see why.
In $\triangle ABC, AD$ is a median and so D is the midpoint of BC.
Therefore, $\triangle ACD$ and $\triangle ABD$ have equal bases ($CD = BD$).
Further, the height of $\triangle ABD$ is equal to the height of $\triangle ACD$ (both are AB).
Thus, $\triangle ACD$ and $\triangle ABD$ have equal bases and equal heights.
Since the area of each triangle equals one-half times the base times the height, then $\triangle ACD$ and $\triangle ABD$ have equal areas and so the median AD divides $\triangle ABC$ into equal areas.
Since $\triangle ABC$ has area 24, then $\triangle ACD$ has area $\frac{12}{2} = 12$.

(b) There are two solutions!!

Solution 1

In $\triangle FSG, FS = 18, SG = 24$, and $\angle FSG = 90°$.
Thus, by the Pythagorean Theorem, $FG = \sqrt{18^2 + 24^2} = \sqrt{324 + 576} = \sqrt{900} = 30$ (since $FG > 0$).
Since, S is on FH so that $FS = 18$ and $SH = 32$, then $FH = FS + SH = 18 + 32 = 50$.
In $\triangle FGH, FH = 50, FG = 30$, and $\angle FGH = 90°$. Thus, by the Pythagorean Theorem,
$GH = \sqrt{50^2 - 30^2} = \sqrt{2500 - 900} = \sqrt{1600} = 40$ (since $GH > 0$).
In $\triangle FGH, FT$ is a median and so T is the midpoint of GH.
In $\triangle FHT$, has $HT = \frac{40}{2} = 20$, and height $FG = 30$. (Since $\angle FGH = 90°, FG$ is the height of $\triangle FHT$ even though FG is outside $\triangle FHT$.)



Thus, $\triangle FHT$ has area $\frac{1}{2}(20)(30) = 300$.

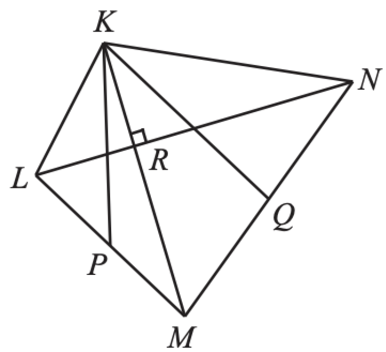
Solution 2 (Uses the fact we found in (a) Solution 3.)

Since, S is on FH so that $FS = 18$ and $SH = 32$, then $FH = FS + SH = 18 + 32 = 50$.
In $\triangle FGH$, base $FH = 50$ and height $SG = 24$ (since SG is perpendicular to FH, SG is a height of $\triangle FGH$).
Thus, $\triangle FGH$ has an area $\frac{1}{2}(50)(24) = 600$.
The median of a triangle divides the area of the triangle in half. (Solution 3 of part (a) shows an example of why a median divides a triangle's area in half.)
Since FT is a median of $\triangle FGH$, then the area of $\triangle FTH = \frac{600}{2} = 300$.

(c) *Solution* (Uses the fact we found in (a) Solution 3 and some more.)

We will use the notation $|\triangle KLM|$ to represent the area of $\triangle KLM, |KPMQ|$ to represent the area of $KPMQ$, and so on.
In $\triangle KLM, KP$ is a median and so $2|\triangle KPM| = |\triangle KLM|$.
(Solution 3 of part a shows an example of why a median divides a triangle's area in half.)

In $\triangle KMN, KQ$ is a median and so $2|KMQ| = |\triangle KMN|$.



Therefore,
$|KLMN| = |\triangle KLM| + |\triangle KMN| = 2|\triangle KPM| + 2|\triangle KMQ|$ and $|KPMQ| = |\triangle KPM| + |\triangle KMQ|$ which tells us $|KLMN| = 2|KPMQ|$
Since $|KPMQ| = 63$, then $|KLMN| = 2|KPMQ| = 2(63) = 126$.
Now $|KLMN| = |\triangle KRL| + |\triangle LRM| + |\triangle KRN| + |\triangle NRM|$.
Each of these four triangles are right-angled.
Since $KR = x$ and $LR = 6$, then $|\triangle KRL| = \frac{1}{2}x(6) = 3x$.
Since $LR = 6$ and $RM = 2x + 2$, then $|\triangle LRM| = \frac{1}{2}(6)(2x + 2) = 6x + 6$.
Since $KR = x$ and $RN = 12$, then $|\triangle KRN| = \frac{1}{2}x(12) = 6x$.
Since $RN = 12$ and $RM = 2x + 2$, then $|\triangle NRM| = \frac{1}{2}(12)(2x + 2) = 12x + 2$.
Therefore,

$$\begin{aligned} |KLMN| &= |\triangle KRL| + |\triangle LRM| + |\triangle KRN| + |\triangle NRM| \\ 126 &= 3x + (6x + 6) + 6x + (12x + 12) \\ 126 &= 27x + 18 \\ 27x &= 108 \\ x &= 4 \end{aligned}$$

Therefore, $x = 4$.

Problem 3 Solution

(a) *Solution*

Since 5 is an odd integer, then n must be an odd integer for the sum $n+5$ to be an even integer.

(b)*Solution*

We first note that the product of an even integer and any other integers, even or odd, is always an even integer.
Let $N = cd(c+d)$.
If c or d is an even integer (or both c and d are even integers), then N is the product of an even integer and some other integers and thus is even.
The only remaining possibility is that both c and d are odd integers.
If c and d are odd integers, then the sum $c+d$ is an even integer.
In this case, N is again the product of an even integer and some other integers and thus is even.
Therefore, for any integers c and d, $cd(c+d)$ is always an even integer.

(c) *Solution*

Since e and f are positive integers so that $ef = 300$, then we may begin by determining the factor pairs of positive integers whose product is 300.
Written as ordered pairs (x, y) with $x < y$, these are:

$$(1, 300), (2, 150), (3, 100), (4, 75), (5, 60), (6, 50), (10, 30), (12, 25), (15, 20).$$

It is also required that that the sum $e+f$ be odd and so exactly one of e or f must be odd.
Therefore, the factor pairs whose sum is odd is

$$(1, 300), (3, 100), (4, 75), (5, 60), (12, 25), (15, 20).$$

Therefore, there are 6 ordered pairs (e, f) satisfying the given conditions.

(d) *Solution*

Since both m and n are positive integers, then $2n > 1$ and so $2n + m > m + 1$.
Let $a = m + 1$ and $b = 2n + m$ or $a = 2n + m$ and $b = m + 1$ so that $ab = 9000$.
We must first determine all factor pairs (a, b) of positive integers whose product is 9000.
We begin by considering the parity (whether each is even or odd) of the factors a and b.

Since 2 is even, then $2n$ is even for all positive integers n.
If m is even then $2n + m$ is even since the sum of two even integers is even.
However if m is even, then $m + 1$ is odd since the sum of an even integer and an odd integer is odd.
That is, if m is even, then a is odd and b is even or a is even and b is odd.
We say that the factors a and b have *different parity* since one is even and one is odd.

If m is odd then $2n+m$ is odd. If m is odd then $m+1$ is even. That is, if m is odd, then a is even and b is odd or a is odd and b is even and so the factors a and b have different parity for all possible values of m.

Now we are searching for all factor pairs (a, b) of positive integers whose product is 9000 with a and b having different parity.

Written as a product of its prime factors, $9000 = 2^3 \times 3^2 \times 5^3$ and so $ab = 2^3 \times 3^2 \times 5^3$

Since exactly one of a or b is odd, then one of them does not have a factor of 2 and so the other must have all factors of 2.

That is, either $a = 2^3 r = 8r$ and $b = s$, or $a = r$ and $b = 8s$ for positive integers r and s. In both cases, $ab = 8rs = 9000$ and so $rs = \frac{9000}{8} = 1125 = 3^2 5^3$.

We now determine all factor pairs (r, s) of positive integers whose product is 1125. These are $(r, s) = (1, 1125), (3, 375), (5, 225), (9, 125), (15, 75), (25, 45)$.

Therefore $(a, b) = (8r, s) = (8, 1125), (24, 375), (40, 225), (72, 125), (120, 75), (200, 45)$, or $(a, b) = (r, 8s) = (1, 9000), (3, 3000), (5, 1800), (9, 1000), (15, 600), (25, 360)$.

Since $2n + m > m + 1 > 1$, then the pair (1, 9000) is not possible. This leaves 11 factor pairs (a, b) such that $ab = 9000$ with a and b having different parity. Each of these 11 factor pairs (a, b) gives an ordered pair (m, n).

To see this, let $m+1$ equal the smaller of a and b, and let $2n+m$ equal the larger (since $2n+m > m+1$).

For example when $(a, b) = (8, 1125)$, then $m + 1 = 8$ or $m = 7$ and so $2n + m = 2n + 7 = 1125$ or $2n = 1118$ or $n = 559$.

That is, the factor pair $(a, b) = (8, 1125)$ corresponds to the ordered pair $(m, n) = (7, 559)$ so that $(m + 1)(2n + m) = 9000$.

Each of the 11 pairs (a, b) gives an ordered pair (m, n) such that $(m + 1)(2n + m) = 9000$. We determine the corresponding ordered pair (m, n) for each (a, b) in the table below (although this work is not necessary since we were only asked for the number of ordered pairs).

(a, b)	$m + 1$	$2n + m$	(m, n)
$(8, 1125)$	8	1125	$(7, 559)$
$(24, 375)$	24	375	$(23, 176)$.
$(40, 225)$	40	225	$(39, 93)$
$(72, 125)$	72	125	$(71, 27)$
$(120, 75)$	75	120	$(74, 23)$
$(200, 45)$	45	200	$(44, 78)$
$(3, 3000)$	3	3000	$(2, 1499)$
$(5, 1800)$	5	1800	$(4, 898)$
$(9, 1000)$	9	1000	$(8, 496)$
$(15, 600)$	15	600	$(14, 293)$
$(25, 360)$	25	360.	$(24, 168)$

There are 11 ordered pairs (m, n) of positive integers satisfying $(m + 1)(2n + m) = 9000$.