Grade 11/12 Math Circles - Fall 2021
Circles, Ellipses, and Astrophysics
Part 1: Fundamentals
SOLUTIONS

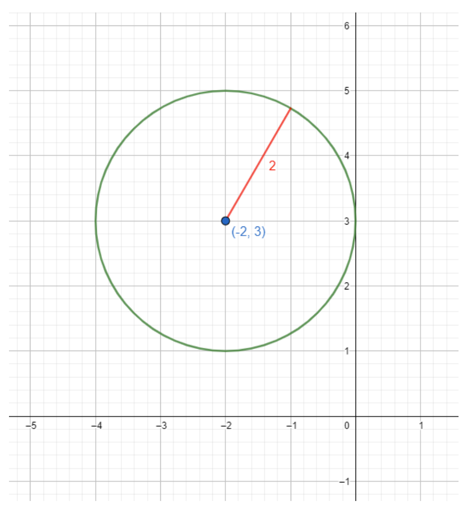
Exercise 1

To draw a circle using these tools, tape one end of the string to the paper and tie your pen/pencil to the other end. Then, keeping the string taut, move the pen/pencil around either clockwise or counterclockwise to trace out a circle. The length of your string will be the circle's radius, and the taped point is the circle's centre.

Exercise 2

(a) Given that the unit circle has radius 1 and centre at the origin, using the equation $(x-a)^2 + (y-b)^2 = r^2$ and substituting $r = 1$, $a = 0$, and $b = 0$, we arrive at $x^2 + y^2 = 1$.

(b) This is the circle with radius 2 and moved left 2 and up 3 (so the centre is at $(-2, 3)$).

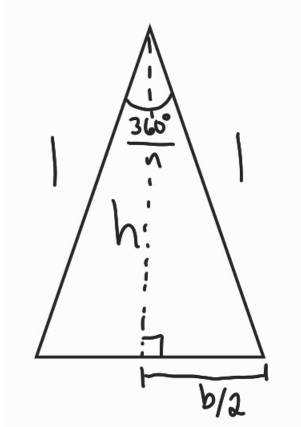


(c) This is the circle with radius 4 and centre at $(4, 4)$. Using the equation $(x-a)^2 + (y-b)^2 = r^2$ and substituting $r = 4$, $a = 4$, and $b = 4$, we arrive at $(x-4)^2 + (y-4)^2 = 16$.

Exercise 3

(a) The total of all the angles in question for all n isosceles triangles $360°$ as this is a full rotation. This angle is split across the n isosceles triangles in each n-gon, meaning the angle in question is $\frac{360°}{n}$.

(b) Consider the diagram on the next page for one of the n isosceles triangles. We can find the base and height of the triangle by constructing a right-angled triangle. We use the fact that $\sin(\theta) = \frac{O}{H}$ and $\cos(\theta) = \frac{A}{H}$, where $H = 1$ for the right-angled triangle in question, and the $\theta = \frac{1}{2}\frac{360°}{n} = \frac{180°}{n}$. We find that $h = \cos(\frac{180°}{n})$ and $\frac{b}{2} = \sin(\frac{180°}{n})$. So, the area of the triangle is $A = \frac{1}{2}(2\sin(\frac{180°}{n}))(\cos(\frac{180°}{n})) = (\sin(\frac{180°}{n}))(\cos(\frac{180°}{n}))$. Using the trigonometric identity provided, we find $A = \frac{1}{2}\sin(\frac{360°}{n})$ as desired.



(c) Each n-gon is made up of n triangles, so we multiply the area found in the previous part by n to arrive at $\frac{n}{2}\sin(\frac{360°}{n})$.

(d) Plugging $n = 8$, $n = 15$, $n = 50$, and $n = 3000$ into the formula in (c) we get $A = 2.828427$, $A = 3.050525$, $A = 3.133331$, and $A = 3.141590$. It is clear the areas are approaching π, which is expecting as the n-gons in question are approaching a unit circle, which has area $A = \pi(1^2) = \pi$.

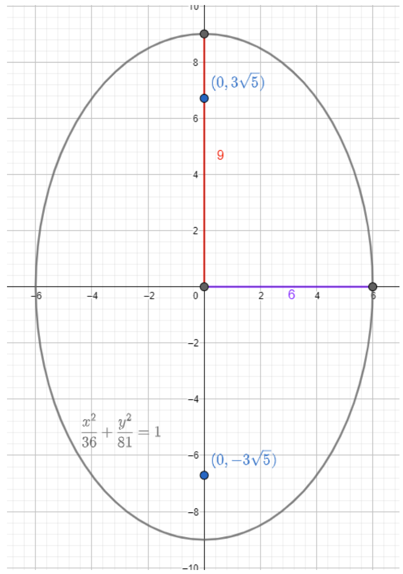
Exercise 4

To draw an ellipse using these tools, tape both ends of the string to the paper at two different places and pull the string taut with your pen/pencil. Then, keeping the string taut, move the pen/pencil around either clockwise or counterclockwise to trace out an ellipse. The length of your string will be the constant distance sum, and the taped point are the ellipse's foci.

Exercise 5

(a) This ellipse has major axis corresponding to the x-axis, where half the length of the major axis is 2 and half the length of the minor axis is 1. Using the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and substituting in $a = 2$ and $b = 1$ we arrive at $\frac{x^2}{4} + y^2 = 1$. Then, substituting $a = 2$ and $b = 1$ into $a^2 = f^2 + b^2$ we find that the foci are at $(\pm\sqrt{3}, 0)$.

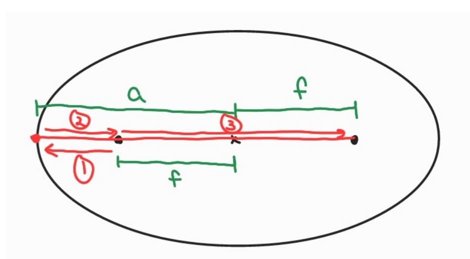
(b) Here, the major axis is along the y-axis. Half the length of the major axis is 9 and half the length of the minor axis is 6. Substituting $a = 9$ and $b = 6$ into $a^2 = f^2 + b^2$ we find that the foci are at $(0, \pm 3\sqrt{5})$.



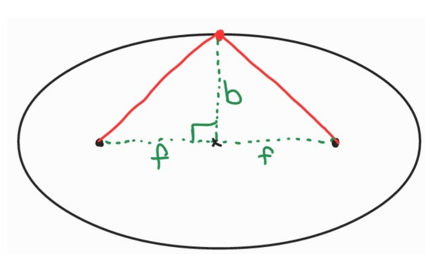
(c) This ellipse has major axis corresponding to the x-axis, where half the length of the major axis is 6 and half the length of the minor axis is not immediately clear. The foci are at $(\pm 3, 0)$. Substituting $a = 6$ and $f = 3$ into $a^2 = f^2 + b^2$ we find that $b = 3\sqrt{3}$. Using the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and substituting in $a = 6$ and $b = 3\sqrt{3}$ we arrive at $\frac{x^2}{36} + \frac{y^2}{27} = 1$.

Exercise 6

(a) In the given pen/pencil position, we can break up the length of the string into three portions (indicated by the arrows in the diagram below). The length of sections 1 and 2 are each $a - f$, and the length of section 3 is $2f$. Thus, the total length of the string is $2(a - f) + 2f = 2a - 2f + 2f = 2a$ as desired.



(b) In the given pen/pencil position, we can draw two right-angled triangles as shown in the diagram below. These two triangles are congruent (SAS), so their hypotenuse must be equal, thus each hypotenuse is exactly half the length of the string, giving us hypotenuse of length a. Then, application of Pythagorean Theorem gives $f = \sqrt{a^2 - b^2}$ as desired.



(c) Here, it is useful to note that the perpendicular line in the diagram intersects the x-axis at $(x, 0)$ and has height y. Now, using the ellipse equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ we can find that $y^2 = b^2(1 - \frac{x^2}{a^2})$. Thus, by applying Pythagorean Theorem on the left side right-angled triangle, we find that

$$\begin{aligned} l_1^2 &= (\sqrt{a^2 - b^2} + x)^2 + y^2 \\ &= (\sqrt{a^2 - b^2} + x)^2 + b^2(1 - \frac{x^2}{a^2}) \\ &= a^2 - b^2 + 2x\sqrt{a^2 - b^2} + x^2 + b^2 - \frac{x^2b^2}{a^2} \\ &= \frac{x^2}{a^2}(a^2 - b^2) + 2x\sqrt{a^2 - b^2} + a^2 \\ &= (\frac{x}{a}\sqrt{a^2 - b^2} + a)^2 \end{aligned}$$

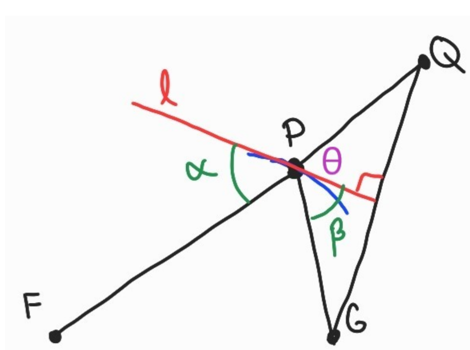
And, applying Pythagorean Theorem on the right side right-angled triangle yields

$$\begin{aligned} l_2^2 &= (\sqrt{a^2 - b^2} - x)^2 + y^2 \\ &= (\sqrt{a^2 - b^2} - x)^2 + b^2(1 - \frac{x^2}{a^2}) \\ &= a^2 - b^2 - 2x\sqrt{a^2 - b^2} + x^2 + b^2 - \frac{x^2b^2}{a^2} \\ &= \frac{x^2}{a^2}(a^2 - b^2) - 2x\sqrt{a^2 - b^2} + a^2 \\ &= (\frac{x}{a}\sqrt{a^2 - b^2} - a)^2 \end{aligned}$$

(d) We have that $l_1 = \pm(\frac{x}{a}\sqrt{a^2 - b^2} + a)$ and $l_2 = \pm(\frac{x}{a}\sqrt{a^2 - b^2} - a)$. For l_1 note that we are adding two positive values, so the result in the brackets is positive, so we take $l_1 = \frac{x}{a}\sqrt{a^2 - b^2} + a$. However, for l_2 note that since $x < a$ we have that $\frac{x}{a} < 1$ and also note that $\sqrt{a^2 - b^2} < \sqrt{a^2}$, so we have that $\frac{x}{a}\sqrt{a^2 - b^2} < a$ and so the result in the brackets is negative. Thus, we take $l_2 = -(\frac{x}{a}\sqrt{a^2 - b^2} - a) = a - \frac{x}{a}\sqrt{a^2 - b^2}$. Finally, this gives us $l_1 + l_2 = \frac{x}{a}\sqrt{a^2 - b^2} + a + a - \frac{x}{a}\sqrt{a^2 - b^2} = 2a$ as desired.

Exercise 7

(a) First, note that by the construction indicated in the question, $\triangle PQG$ is isosceles with $PQ = PG$. Introduce angle θ as shown on the diagram below. Then, as l is the perpendicular bisector, we have split the isosceles triangle into two congruent right-angled triangles (SSS). Thus we have that $\theta = \beta$. Now since θ and α are opposite angles, we have that $\alpha = \theta$. Thus, we have $\alpha = \beta$ as desired.



(b) As l is a perpendicular bisector, it divides QG into half. Thus, we have two right-angled triangles whose base (half of QG) and height (the length of l from QG to point R) are the same. Thus, by Pythagorean Theorem, we have that $RQ = RG$.

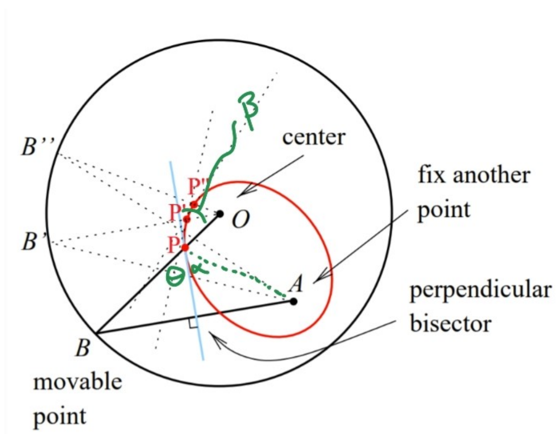
Now, recall that $PG = PQ$, and thus $FQ = PF + PQ = PF + PG$. Now, examining $\triangle FRQ$, by the triangle inequality we have $RF + RQ \geq FQ$ with equality only when R sits on FQ. That is, since $RQ = RG$ and $FQ = PF + PG$ that we have $RF + RG \geq PF + PG$ as desired.

Note that the argument here is that $PF + PG$ is the sum of the distance from the point P to the foci. By definition, for an ellipse, this value is constant for any point along the ellipse. But we have found $RF + RG > PF + PG$, so R cannot be a different location along the ellipse (we do have the case of equality, which only holds when R and Q are the same point). So, l cannot intersect the ellipse at another point, and is thus tangent to the ellipse at P.

Exercise 8

(a) First, note that the radius of the circle, OB, is by definition constant. Now, consider $\triangle PBA$. The perpendicular bisector cuts AB into half, thus giving us two right angled triangles that are congruent (SAS). This means that $PA = PB$. Thus, we have that $OB = OP + PB = OP + PA$ is constant, as required.

(b) To show the perpendicular bisector is a line of type l, we must show that $\alpha = \beta$ as indicated on the diagram below. Since we had that the two right-angled triangles are congruent from the previous part, introduce θ, which due to congruency has $\theta = \alpha$. Now, since θ and β are opposite angles, we have $\theta = \beta$, thus giving $\alpha = \beta$ as desired.



Exercise 9

Since the distances must have been travelled over the same span of time, the planet must be orbitting faster when it was closer to the Sun than when it was further from the Sun in order to travel a further distance. This makes sense from a physical perspective, as the gravitational force on the planet from the Sun is stronger when the planet is closer.

Exercise 10

(a) Returning to our ellipse equation symbolism and using the diagram showing where perihelion is, we are given that $a = 5.203$ and that $a - f = 4.950$. We can use these to find that $f = 0.253$. Then, plugging into the equation eccentricity $= \frac{f}{a} = \frac{0.253}{5.203} = 0.0486$.

(b) Referencing the diagram showing where aphelion is, we have that the aphelion distance from the Sun is $a + f$. Plugging in $a = 5.203$ and $f = 0.253$ gives that Jupiter's aphelion distance is 5.456 AU.

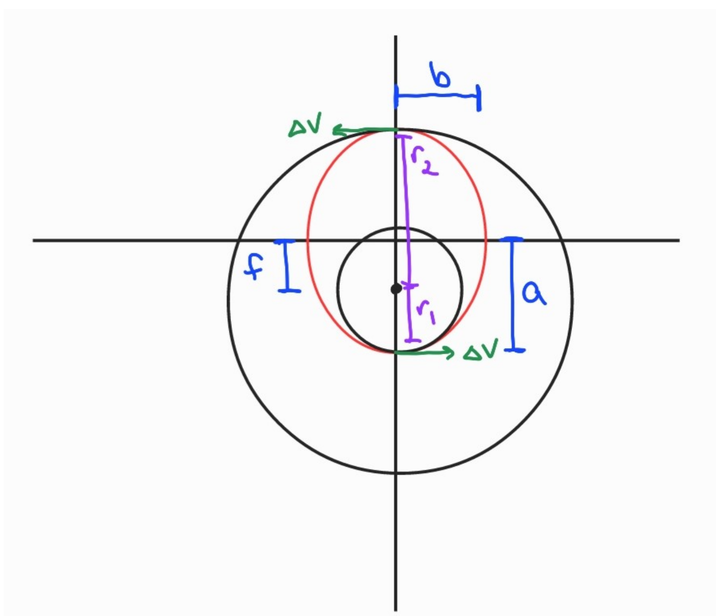
(c) Using Kepler's Third Law in the context of our own solar system, $P^2 = a^3$, and substituting $a = 5.203$ gives that $P = 11.87$ years.

(d) From Exercise 9, Jupiter is moving fastest at perihelion and slowest at aphelion. Thus Jupiter moves slowest at a distance of 5.456 AU and fastest at a distance of 4.950 AU from the Sun.

(e) As we are setting up the major axis along the x-axis, we make use of the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. We have $a = 5.203$. We will need to find b using $a^2 = f^2 + b^2$. Substituting in $a = 5.203$ and $f = 0.253$ gives $b = 5.197$. Thus, the equation of Jupiter's elliptical orbit in this construction is $\frac{x^2}{5.203^2} + \frac{y^2}{5.197^2} = 1$.

Exercise 11

(a) Drawing the situation, as below, for the Hohmann transfer ellipse with semimajor axis along the y-axis of length a, semiminor axis length b, focus length f, and two circular orbits of radii r_1 and r_2, provides some key relationships. First, note that $2a = r_1 + r_2$. Next, note that $f = a - r_1$. We can then find b once we find a and f.



Plugging in $r_1 = 21000$ and $r_2 = 42300$ into $2a = r_1 + r_2$, we find that $a = 31650$. Then, plugging in $a = 31650$ and $r_1 = 21000$ into $f = a - r_1$, we find that $f = 10650$. Then, subbing

$a = 31650$ and $f = 10650$ into $a^2 = f^2 + b^2$ we find that $b = 29804$.

Finally, substituting into the ellipse equation of form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, we get the equation of the Hohmann transfer's elliptial orbit is $\frac{x^2}{29804^2} + \frac{y^2}{31650^2} = 1$.

For the circular orbits, note that both have centres at $(0, -f)$ and the radii are r_1 and r_2. Thus the initial circular orbit has equation $x^2 + (y + 10650)^2 = 21000^2$ and the final circular orbit has equation $x^2 + (y + 10650)^2 = 42300^2$.

(b) We have the same relationships for the Hohmann transfer ellipse as in the previous problem, except now with semimajor axis along the x-axis.

Plugging in $r_1 = 1$ and $r_2 = 1.52$ into $2a = r_1 + r_2$, we find that $a = 1.26$. Then, plugging in $a = 1.26$ and $r_1 = 1$ into $f = a - r_1$, we find that $f = 0.26$. Then, subbing $a = 1.26$ and $f = 0.26$ into $a^2 = f^2 + b^2$ we find that $b = 1.23$.

Finally, substituting into the ellipse equation of form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get the equation of the Hohmann transfer's elliptial orbit is $\frac{x^2}{1.26^2} + \frac{y^2}{1.23^2} = 1$.

For the circular orbits, note that both have centres at $(-f, 0)$ and the radii are r_1 and r_2. Thus the initial circular orbit has equation $(x + 0.26)^2 + y^2 = 1^2$ and the final circular orbit has equation $(x + 0.26)^2 + y^2 = 1.52^2$.