Grade 11/12 Math Circles - Fall 2021
Circles, Ellipses, and Astrophysics
Part 2: A Focus(!) on Kepler

Recall in the previous lesson, we introduced Kepler's Three Laws of Planetary Motion:

1. All planets move in elliptical orbits, with the sun at one focus.

2. The line connecting a planet to the sun sweeps out equal areas in equal times.

3. The square of the period of the orbit of any planet is equal to the cube of the semimajor axis of its orbit times a constant (we say they are proportional to each other, or $P^2 \propto a^3$). For the planets in our system revolving around the Sun, using units of AU (astronomical units - where 1 AU is the Sun to Earth distance) for distance and years for time, this constant is 1, giving rise to the formula $P^2 = a^3$

Newton, one of the founding figures of calculus, was the first to provide purely geometric proofs of the first two laws. For the third law, we will satisfy ourselves with a simplified case and some empirical investigation. Much later, famed physicist and educator Richard Feynman adapted Newton's proof into his own, slightly simpler, version of the proof. This famous version of the proof of Kepler's first law in particular is known as "Feynman's Lost Lecture" thanks to the fact it was never properly recorded nor written down. A large part of this second lesson is an adaption of Hall & Higson's (1998) and Tong's (2012) distillations of this lost lecture.

Some Background Physics

Before we can get to our geometric and empirical proofs, we must introduce several pieces of physics knowledge. The bulk of the content in this section tends to be covered in a first physics course, and the derivations (which are more physical than mathematical in nature) are widely available, for the curious reader. We do not focus on these derivations, but instead take these as fundamental facts to be used as tools in our proofs later on.

We will be making use of vectors in this lesson. Symbolically, we indicate a vector by putting a little arrow over top of the letter(s), as such: $\vec{v}$, or by bolding the letter(s), as such: $\mathbf{v}$. A vector represents a magnitude (size) and direction. We draw a vector as an arrow, with the arrowhead (tip) pointing in the direction of interest, and the length of the arrow indicating the magnitude. We notate the vector's length, or norm, as $||\vec{v}||$. We can add two vectors $\vec{v}$ and $\vec{u}$ emanating from the same point by completing the parallelogram they build, as shown below.

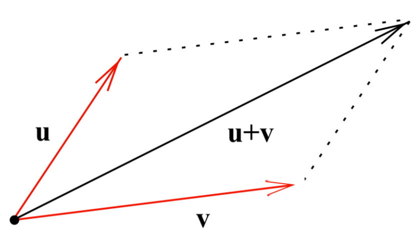


Image credit: Hall & Higson

Just as there are three famed laws of planetary motion named after Kepler, there are three famed laws of motion in classical mechanics named after Newton: Newton's Three Laws of Motion. We state here the first two laws, as they will come in useful for proving Kepler's Second and Third Laws.

1. Unless acted upon by an external force, an object at rest will stay at rest, and an object in motion will continue travelling in a straight line at constant speed.

2. The change in motion of an object is proportional to and in the direction of any force acting on the object. This law can also be restructured to provide the famous formula $F = ma$, which tells us that the total force on an object is equal to its mass times its acceleration.

Newton is also known for his famous inverse-square law, which we will use in proving Kepler's First Law. The inverse-square law states that the change in acceleration of a planet is proportional to the inverse square of its distance (here, we take a circular approximation, which is relatively fair, given our discussion on the real-life eccentricities of planetary orbits in the first lesson):

$$||\Delta\vec{a}|| = \left\|\frac{\Delta\vec{v}}{\Delta t}\right\| = \frac{\text{constant}}{r^2}$$

As if this wasn't enough for Newton's contributions, we will also make use of Newton's Law of Universal Gravitation, which describes the gravitational force between two objects of masses M and m, when we make our way towards deriving a simplified case of Kepler's Third Law:

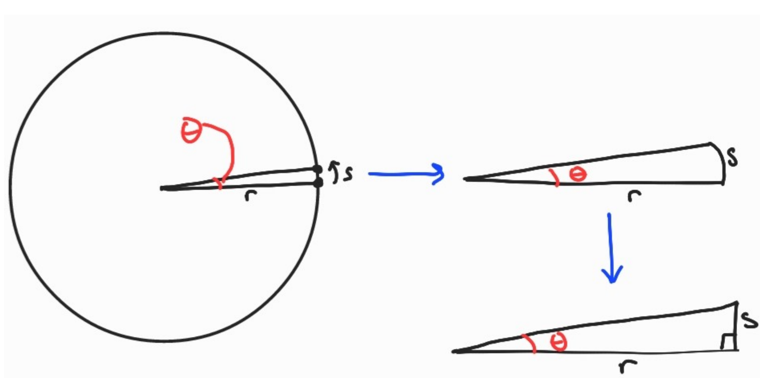
$$F_g = \frac{GMm}{r^2}$$

where G is the gravitational constant, and has value $G \approx 6.67 \times 10^{-11}\ \text{m}^3\text{kg}^{-1}\text{s}^{-2}$.

Further, we introduce here a few key facts for an object in circular motion. Firstly, at any given point in its circular motion, the object's velocity vector is tangent to the circular motion, and its acceleration vector points directly towards the centre of the circle (and is called thusly the centripetal acceleration), and has value:

$$a_c = \frac{v^2}{r}$$

Lastly, we introduce here a common approximation in the realm of astrophysics: the small angle approximation. Consider the massive scales of planetary orbits, which are nearly circular, and imagine a planet has move a little ways along its orbit. The central angle (θ) that it has conversed is incredibly small, and the path it has travelled, while technically an arc (of length $s = r\theta$), can be approximated as a straight line, as in the diagram below. In our context, we thus approximate motions of planets through small angles as straight lines, rather than as arcs.



Proof of Kepler's Second Law

Let us suppose a planet travels from A to B in a certain unit of time (taking the small angle approximation, we take it to have travelled in a straight line). If no force were acting on the planet, by Newton's First Law, the planet would continue onward to point c after another unit of time has elapsed.

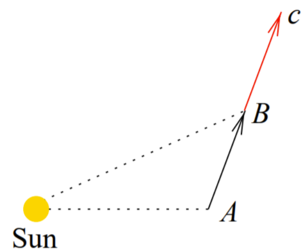


Image credit: Hall & Higson

However, thanks to the Sun, there is in fact an external force acting on the planet: gravity! The force vector of gravity on the planet by the Sun points directly towards the Sun, as the Sun pulls the planet in towards it. Then, the planet does not actually arrive at the point c, but at a point C closer towards the Sun. This closer point C is found by taking the vector sum of the the motion there would be with no external force and the force of gravity. We continue this process along the planet's path to find its new arrival points D, E, etc. after each **equal unit time step**.

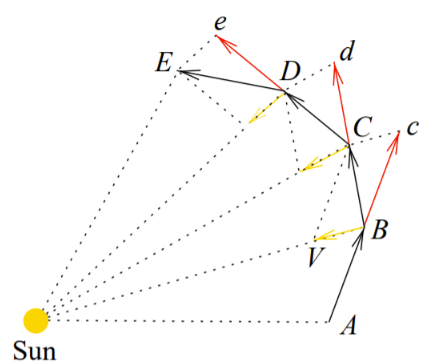


Image credit: Hall & Higson

Exercise 1

Consider the previous diagram. Call the point where the Sun is S. Focus on $\triangle SBC$ and $\triangle SCD$. Extend their common side SC, and draw two parallel lines to this extension, one extending from B, and the other from D.

(a) Argue why C is the midpoint of Bd.

(b) Show that the distances between the parallel lines you were asked to construct are equal.

(c) Show that the areas of $\triangle SBC$ and $\triangle SCD$ are equal.

(d) Argue why this proves Kepler's Second Law.

Proof of Kepler's First Law

Recall the main thrust of this proof: we want to show a planet's orbit is elliptical.

Feynman's big idea for this proof was to divide up a planet's orbit into **equal angle** segments (rather than equal time segments, as Newton did).

Exercise 2

Using the small angle approximation, show that area swept out by a planet that has traversed through small angle θ is (approximately) proportional to the square of the orbital radius. That is, show that $A \propto r^2$ (or equivalently $A = \text{constant} \times r^2$).

Throughout this proof, we will concern ourselves with the velocity vector of a planet's motion. The first key that we want to unlock is that the planet's change in velocity with respect to change in angle is constant.

Exercise 3

Let Δt be the time it takes the planet to traverse θ. Let $\Delta \vec{v}$ be the change in the planet's velocity over Δt, and take note that $\Delta \vec{v}$ will point towards the Sun.

(a) Use Newton's inverse square law to find an expression for $||\Delta \vec{v}||$.

(b) Use Exercise 2 to further the expression in (a).

(c) Use Kepler's Second Law to show that $||\Delta \vec{v}|| = \text{constant}$. Thus, we have shown that the planet's change in velocity with respect to change in angle is constant.

Now, we want to construct a velocity diagram. To do this, we break up the orbit into equal angle pieces, and draw the velocity vectors at the start of each segment. We then take all of the velocity vectors and move them so their tails all meet to create a velocity diagram. This is shown below:

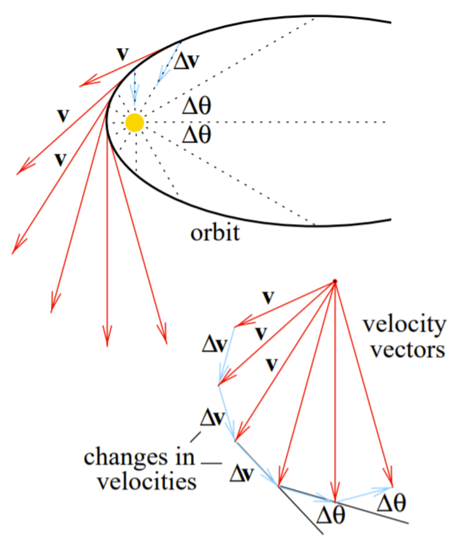


Image credit: Hall & Higson

Exercise 4

(a) Explain why all of the $\Delta\vec{v}$ vectors are the same length.

(b) Explain why the $\vec{v}$ vectors have differing lengths. Where in an orbit would they be the longest? The shortest?

(c) Show why each $\Delta\vec{v}$ points $\Delta\theta$ beyond the previous one in the velocity diagram.

If we completed the velocity diagram, we would find that it creates a regular polygon, as shown below:

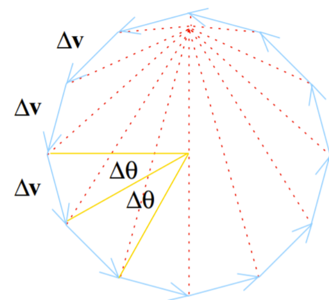


Image credit: Hall & Higson

Exercise 5

(a) What is the side length of the regular polygon?

(b) How many sides does the regular polygon have?

(c) Use the previous fact to show, as indicated in the diagram, that the angle created between the ends of each vector and the **centre** of the polygon is $\Delta\theta$. Note that this means that the angle swept from the centre of the velocity diagram is equal to the angle the planet sweeps out from the sun!

(d) As $\Delta\theta$ gets smaller and smaller, what shape does the regular polygon approach?

Now, we must go from the shape of the velocity diagram (in the limit as $\Delta\theta$ gets smaller and smaller) to the shape of the actual orbit. A key here is that the velocity vectors to the planet's orbit are all tangent to its orbit. Let's focus on one such tangent to the orbit (of as yet unknown shape) at point P, occurring after sweeping out an angle θ, which we call l as shown below:

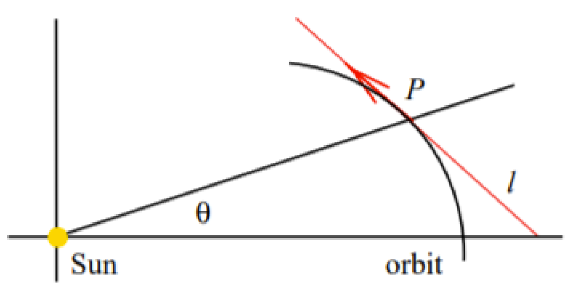


Image credit: Hall & Higson

Note l would be in the same direction as the velocity vector at P, so we can on our velocity diagram for the orbit, locate line l' which lies along that velocity vector and is thus parallel to l, as shown below:

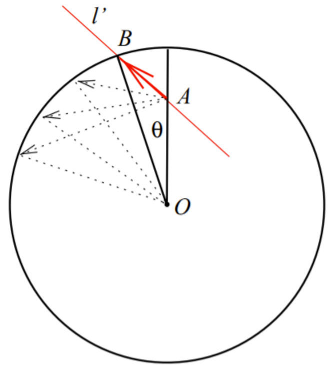


Image credit: Hall & Higson

In this diagram, we let B be the intersection of line l' with the circle (the shape of our velocity diagram as θ gets very small), and let O be the centre of the circle. Recall we found in Exercise 5 that the angle swept out in the orbit is equal to the angle swept out from the centre of the velocity diagram, so we have that $\angle AOB = \theta$. In the diagram, A is the point from which all of the velocity vectors in the diagram sprout.

Now, the next wild part. Rotate this velocity diagram 90° clockwise. Note that this makes l' perpendicular to l. Now, construct the perpendicular bisector, p, to AB. This is shown below:

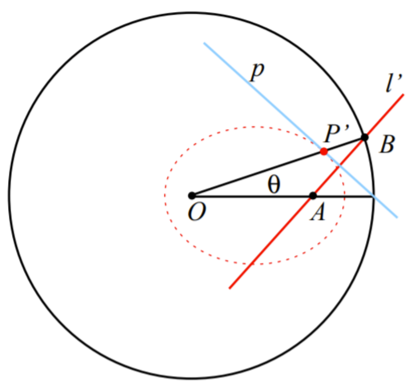


Image credit: Hall & Higson

Exercise 6

Hopefully, the construction we just did reminds you of something from lesson 1. If it doesn't, go back and skim that lesson before proceeding!

(a) Argue that the point P', which is the intersection of p with OB, lies on an ellipse.

(b) If p is the perpendicular bisector to l' in this diagram, how is p related to l?

What we have just discovered is that for every angle θ, the tangent line to our original orbit is tangent to an ellipse. If two curves have the same tangents at every single point, they must be the same curve. Thus, a planet's orbit is elliptical.

I invite you to check out this wonderful video that walks you through the proof we just did, bringing together lesson 1 and 2: `https://www.youtube.com/watch?v=xdIjYBtnvZU`.

Proof of Kepler's Third Law and an Extension

For this final law, after spending a great deal of energy on proving that orbits are elliptical, we are going to approximate elliptical orbits to be circular (sorry!) to show a simplified case.

Exercise 7

Consider a planet in a circular orbit. This would mean that the semimajor axis length is the radius ($a = r$).

(a) Use Newton's second law and the formula for centripetal acceleration to come up with a formula for the net force on a planet in a circular orbit.

(b) If the only force acting on the planet is the gravitational force between the Sun and the planet, use Newton's Law of Universal Gravitation in conjunction with part (a) to find a find a formula for v^2.

(c) The period of an orbit is the length of time it takes to complete one complete orbit. Argue why this gives $P = \frac{2\pi r}{v}$.

(d) Use (c) in conjunction with (b) to show that $P^2 \propto r^3$. This complete the proof for a circular orbit.

Exercise 8

Below is a table of data that was used by Kepler in 1618. Use this data to empirically arrive at Kepler's Third Law.

Planet	Mean distance to sun (AU)	Period (days)
Mercury	0.389	87.77
Venus	0.724	224.70
Earth	1	365.25
Mars	1.524	686.95
Jupiter	5.20	4332.62
Saturn	9.510	10759.2

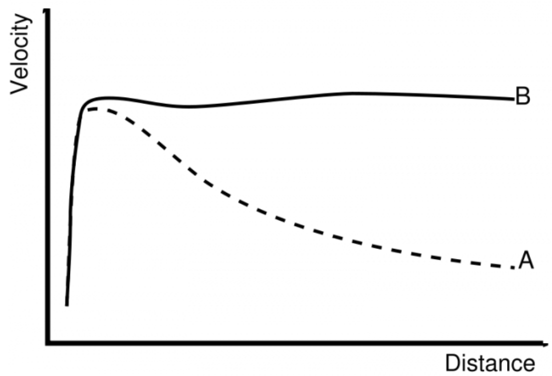
Exercise 9

We can extend our work now to some interesting physics. Consider the formula you found in Exercise 7b. We can use the data in Exercise 8 and this formula to find the orbital velocity of the planets in the solar system. Construct a scatter plot of planetary distance from the Sun in AU (on the x-axis) versus orbital velocity in km/s (on the y-axis). You will need to use the value of the gravitational constant (given earlier in the lesson), the conversion 1 AU $= 1.496 \times 10^8$ km, and the mass of the Sun, which is 2×10^{30} kg.

The shape of the curve which connects the dots in the scatter plot you created in Exercise 9 is known as a Keplerian rotation curve. Any system with a massive central body and orbitting bodies is expected to follow some kind of modified Keplerian rotation curve.

This applies then not only to any star system, but also to galaxies. Galaxies are composed of gas and stars (in the innermost regions) orbitting around a supermassive central black hole. Thus, when we first set out to explore galaxies outside of our own, we had an expectation of what we would find the rotation curves of these galaxies to look like.

Below is a schematic of what we had expected to find (A) versus what we actually discovered (B) when we looked at other galaxies.



You can see that our expectation was a modified Keplerian rotation curve (with an initial spike due to the mass towards the centre regions of a galaxy), but what we actually found was that galactic rotation curves are largely flat (they do not fall!).

Exercise 10

Consider once again your formula from Exercise 7b. This time though, consider the big mass, M, to be the total mass contained within whatever radius you are at in a galaxy (rather than considering it as the mass of the Sun).

(a) Consider our expectation curve. What did we expect to find in terms of the mass and radius as we moved outwards in a galaxy?

(b) Consider the curve we actually found. What does this mean then in terms of the actual mass distribution of a galaxy? This realization is what came to be known as dark matter.