



Grade 6 Math Circles

October 6, 2021

Irrational Numbers

Introduction

We encounter many different numbers in everyday life. Mathematicians from long ago also worked with numbers, and they came up with a way to categorize them based on various properties. For example, we define the set of **integers** to be the set of positive whole numbers, negative whole numbers, and 0. i.e. $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. We use curly brackets to denote a set, which is simply a collection of objects in no particular order. It is common to represent the set of integers with the symbol \mathbb{Z} .

We also define the set of **real numbers** as any number that can be represented as a point on a number line. By this definition, every integer is also a real number, since we can plot the integers on a number line. One example of a non-real number is infinity, since we cannot represent infinity by a point on a line, even if the line is infinitely long. Real numbers are typically represented by the symbol \mathbb{R} .

In this lesson, we would like to talk about the set of irrational numbers. However, before we can define what an irrational number is, we must first define the set of rational numbers.

Rational Numbers

A **rational number** is a number that can be represented as a fraction with an integer numerator and a non-zero integer denominator. Recall that the numerator of a fraction is the number on top of the horizontal line, and the denominator is the number below the line. You can also think of these numbers as ratios, which is where the word “rational” comes from. For example, $\frac{1}{2}$, $\frac{7}{9}$, $\frac{14}{3}$, and -157 are all rational numbers.

We use the symbol \mathbb{Q} to denote the set of rational numbers.

Stop and Think

Why is -157 a rational number? Can you come up with 5 different rational numbers?



Irrational Numbers

An **irrational number** is a real number that is not rational. In other words, an irrational number is a number that cannot be written as a fraction with an integer numerator and a non-zero integer denominator. One interesting property of irrational numbers is that if you convert an irrational number into a decimal, the decimals will neither terminate nor repeat.

The concept of irrational numbers was discovered by a Pythagorean philosopher called Hippasus at around 500 BC. However, other philosophers found his theory so ridiculous that he was thrown into the sea as punishment. Today, irrational numbers are used by mathematicians all over the world, and luckily for us, nobody will be thrown into the sea.

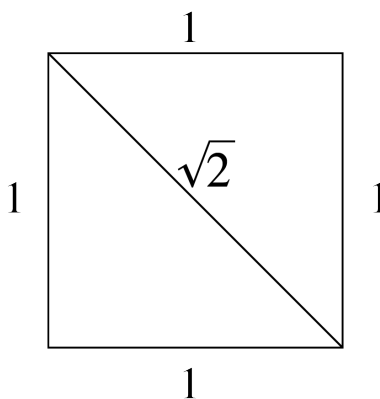
Stop and Think

Use a calculator to convert the fractions $\frac{1}{2}$ and $\frac{4}{3}$ into decimal numbers. What do you notice?

Noteworthy Irrational Numbers

Square Root of 2

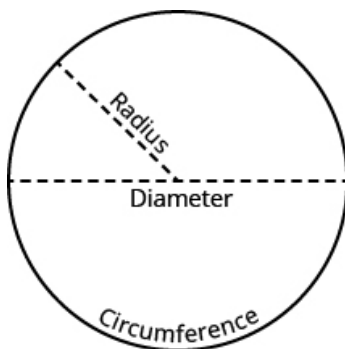
$\sqrt{2} \approx 1.4142\dots$ was likely the first irrational number that Hippasus worked with. One geometrical representation of $\sqrt{2}$ is that it is the length of a diagonal across a unit square (a square with a side length of one unit). Can you verify this using the Pythagorean Theorem?





Pi

$\pi \approx 3.14159\dots$ is a very famous mathematical constant. Geometrically, it represents the ratio of a circle's circumference to its diameter.



π is used widely in all areas of mathematics and physics.

The Golden Ratio

The golden ratio is usually represented using the Greek letter *phi* (ϕ), and is approximately equal to 1.618033.... If the ratio of two numbers a and b is equal to the ratio of their sum and the larger of the two numbers, then we say that $\frac{a}{b}$ is the golden ratio. Algebraically, for two numbers a and b with $a > b > 0$, if

$$\frac{a+b}{a} = \frac{a}{b},$$

then

$$\frac{a}{b} = \phi$$

The golden ratio is often used to analyze proportions of objects in design and architecture. It also appears in some patterns in nature, such as the spiral arrangement of leaves.

Exercise 1

Identify the irrational number(s) from the options below.

(a) $\sqrt{8}$

(b) 2021.1006

(c) $\frac{-79}{1084}$

(d) $\sqrt{9}$

(e) $\frac{0}{\sqrt{2}}$

The set of irrational numbers, combined with the set of rational numbers, make up the set of **real numbers**. Since there is no universal symbol for the set of irrational numbers, we can use $\mathbb{R} - \mathbb{Q}$ to represent the set of real numbers that are not rational.



Continued Fractions

You might be wondering, if π is an irrational number with an infinite number of decimals, why do people say that π is equal to 3.14? Well, this statement is technically not correct. It is better to say that π is *approximately* equal to 3.14, since we have rounded the decimal portion to two places. When doing calculations, you may be using the π button on your calculator, but since your calculator cannot display an infinite number of digits, even your calculator is using an approximation.

Irrational numbers are typically hard to work with in calculations. Therefore, we would like to approximate irrational numbers with rational numbers. You may have heard that $\frac{22}{7} = 3.142857142857\dots$ is a rational approximation for π . It is a good approximation for π , but can we come up with a more accurate rational approximation? The answer lies in continued fractions.

Reciprocals

Before we talk about continued fractions, we need to talk about reciprocals. The **reciprocal** of a number n is $\frac{1}{n}$. Taking a reciprocal of a number is like “flipping the number over”. For example, we know that the number 2 is equal to $\frac{2}{1}$, and when we “flip it over”, we see that the reciprocal of 2 is $\frac{1}{2}$. Similarly for fractions, the reciprocal of a fraction $\frac{a}{b}$ is $\frac{b}{a}$.

Exercise 2

Find the reciprocal of the following numbers. Express your answer as an integer or fraction.

(a) $\frac{1}{5}$

(b) $\frac{17}{36}$

(c) $\frac{-1}{10}$

(d) 36.08

(e) 0.75681

When we take the reciprocal of any number other than 1 and -1 , the value of the number changes. For example, the reciprocal of 2 is $\frac{1}{2}$, and they are not equal. The way to make the two numbers equal again is to take the reciprocal one more time: turn the upside-down number right side up again. So we have $\frac{1}{1/2} = \frac{1}{0.5} = 2$. We will be using this method to create our continued fractions.

Continued Fraction Representation of π

The best way to illustrate continued fractions is with an example, so let us try to find the continued fraction representation for π .

$\pi = 3.14159265\dots = 3 +$ “a little bit more”. Now we are going to take the reciprocal of the “little



bit more” twice.

$0.14159265\dots = \frac{0.14159265\dots}{1}$, and when we take the reciprocal twice and simplify the denominator using a calculator, the “little bit more” becomes

$$\frac{1}{\frac{1}{0.14159265\dots}} = \frac{1}{7.06251348\dots}$$

So,

$$\pi = 3 + 0.14159265\dots = 3 + \frac{1}{7.06251348\dots} = 3 + \frac{1}{7 + \dots}$$

If we stopped here, we can create a rational approximation for π by cutting off the decimal portion of $7.06251348\dots$. If we round to 7, then π is now approximately equal to $3 + \frac{1}{7} = \frac{22}{7}$, which is the rational approximation we talked about earlier.

Let us continue the algorithm in order to find a more accurate rational approximation. The number $7.06251348\dots$ is $7 +$ “a little bit more”, so we once again take the reciprocal of the “little bit more” twice.

$$0.06251348\dots = \frac{1}{\frac{1}{0.06251348\dots}} = \frac{1}{15.99654986\dots}$$

Therefore,

$$7.06251348\dots = 7 + \frac{1}{15.99654986\dots}$$

Putting everything together,

$$\pi = 3 + \frac{1}{7.06251348\dots} = 3 + \frac{1}{7 + \frac{1}{15.99654986\dots}} = 3 + \frac{1}{7 + \frac{1}{15 + \dots}}$$

Here, we are essentially “continuing” the denominator of the fraction by representing the decimal portion as a mixed fraction.

If we stop the algorithm at this step, we have another rational approximation for π .

$$\pi \approx 3 + \frac{1}{\left(7 + \frac{1}{15}\right)} = 3 + \frac{1}{\left(\frac{106}{15}\right)} = 3 + \frac{15}{106} = \frac{333}{106} = 3.14150943396\dots$$



which is much more accurate than $\frac{22}{7}$.

Exercise 3

Continue the algorithm one more step and find a more accurate rational approximation for π .

Finite and Infinite Continued Fractions

You may be wondering, when does this algorithm stop? It may be surprising to find that it doesn't! This is because π is an irrational number, so it has infinitely many non-repeating decimals. If the algorithm stopped at some point (the “little bit more” is no longer there), that would make π rational! Therefore, the continued fraction of π goes on forever.

By this property of irrational numbers, we can see that all irrational numbers have infinite continued fractions. Since we cannot physically compute the entire continued fraction, we use an ellipsis (...) to denote that there is still more to the fraction than what we're putting on the page. In contrast, the continued fraction algorithm does terminate for rational numbers. For example, the continued fraction representation for the number 4.357 is

$$4.357 = 4 + \frac{1}{2 + \frac{1}{1 + \frac{1}{4 + \frac{1}{35 + \frac{1}{2}}}}}$$

Notice that this continued fraction is not approximating 4.357, instead they are exactly equal. Hence, we do not use an ellipsis here. This is also another way to convert a rational number in decimal form to a fraction.

Exercise 4

Convert 4.357 into an improper fraction (an improper fraction is a fraction with the numerator bigger than the denominator) by using its continued fraction above. Can you convert 4.357 to an improper fraction in a different way?



A New Notation

You may have noticed that continued fractions are tedious to write and take up a lot of space on the page. To illustrate this, take a look at the continued fraction representation of an important irrational mathematical constant: Euler's (*pronounced Oiler*) number: $e \approx 2.71828\dots$

$$e = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1 + \frac{1}{1 + \frac{1}{6 + \ddots}}}}}}}}}$$

Notice that the numerator of all the fractions striking down to the right are all 1, so it is redundant to keep writing them out. Instead, we can use a new notation and represent the fraction by listing the denominators in order. In this case, we can write

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, \dots]$$

Notice that we use a semicolon after the first number instead of a comma. This is to distinguish the first number (which is not part of the fraction) from the others.



To generalize, suppose the continued fraction of a number x is as follows (here, we use letters with subscripts to denote arbitrary values, and to show that these values must be in order from 0 to n).

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\ddots + \frac{1}{a_n + \ddots}}}}}$$

We can instead write

$$x = [a_0; a_1, a_2, a_3, \dots, a_n, \dots]$$

The ellipsis in both notations signify that we are omitting/leaving out information. For example, we know there will be numbers between a_3 and a_n , but since this is a general formula, we don't know exactly how many numbers there will be, so we leave out the information for simplicity and generalization.

If we want to stop at step n of the algorithm and provide an estimation for x instead, then we write

$$x \approx [a_0; a_1, a_2, a_3, \dots, a_n]$$

Since we are now using this rational value to estimate x , we say that this is the n^{th} **convergent** to x .

Exercise 5: Putting it all together

Compute the continued fraction representation of $\sqrt{2}$ up to the third step, its rational approximation to $\sqrt{2}$ at that step (as an improper fraction), and the 3rd convergent to $\sqrt{2}$ (using the simplified notation). Do you think this is a good approximation for $\sqrt{2}$?