# Grade 7/8 Math Circles
## October 6th, 2021
## Probability

## Introduction

A **probability** is the likelihood of something happening. It is commonly represented by a percentage, like 25% or 50%, which can also be written as a decimal, like 0.25 or 0.5. For simplicity, we often write the probability of events as a decimal (or fraction) between 0 and 1, where 0 represents certainty the event will not occur (0%) and 1 represents certainty the event will occur (100%).

An **event** is an outcome or result of an experiment to which we assign a probability. Examples of events are getting heads in a coin flip, rolling a 7 with dice, or drawing an Ace from a deck of cards.

For an event $A$, we denote the probability of $A$ by $P(A)$. One way to calculate the probability of an event is to divide the number of outcomes where the event occurs, denoted by $|A|$, by the total number of possible outcomes, denoted by $S$. This gives the following formulas:

$$P(A) = \frac{|A|}{S} \quad \text{or} \quad |A| = S \times P(A) \quad \text{or} \quad S = \frac{|A|}{P(A)}$$

**Example 1**

In a standard deck of 52 cards there are 4 possible suits: **Hearts**, **Diamonds**, **Clubs** and **Spades**; and 13 possible values: **Ace**, **2**, **3**, **4**, **5**, **6**, **7**, **8**, **9**, **10**, **Jack**, **Queen** and **King**. For each suit, each of the 13 values occurs exactly once.

Suppose we randomly draw a card from the deck. What is the probability that the card is a **Club**?

**Solution 1**

There are 52 cards total, so $S = 52$, and 13 **Clubs** in the deck, so $|A| = 13$ . So, the probability of the card being a **Club**, or $P(A)$, is $\frac{|A|}{S} = \frac{13}{52} = 0.25$ or 25%.

**Activity 1**

Use the three formulas listed above to answer the following questions.

(a) What is the probability of rolling an even number on a standard 6-sided die ('die' is singular for 'dice')?

(b) What is the probability of drawing a face-card (Jack, Queen, King) from a standard deck of 52 cards?

(c) How many times does the event $A$ occur if out of 25 total outcomes, $A$ has a probability of 0.64?

(d) How many total outcomes are there if the event $B$ occurs 54 times and has a probability of 0.4?

The **complement** of an event $A$ is denoted by $\overline{A}$ and defined as every outcome that is not in $A$. (i.e. for a coin flip if $A$ is 'heads' then $\overline{A}$ is 'tails'). This gives us the following formulas:

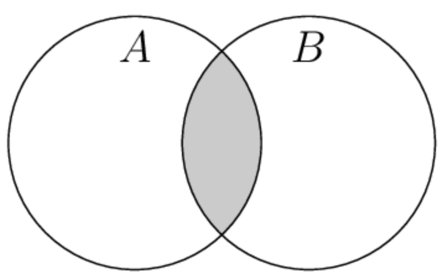
$$|\overline{A}| = S - |A| \quad \text{or} \quad |A| = S - |\overline{A}| \quad \text{or} \quad |A| + |\overline{A}| = S$$

and

$$P(\overline{A}) = 1 - P(A) \quad \text{or} \quad P(A) = 1 - P(\overline{A}) \quad \text{or} \quad P(A) + P(\overline{A}) = 1$$

## Intersection of Events

It is often useful to find the probability of two events $A$ and $B$ both occurring. We call this the **intersection** of $A$ and $B$, and it is denoted by $A \cap B$ or $B \cap A$. If we were to represent events $A$ and $B$ as a Venn Diagram, then $A \cap B$ would be the overlap or shaded region seen below.




So $A \cap B$ occurs only when both events $A$ and $B$ have occurred, meaning that $A \cap B$ can't occur if neither or only one of $A$ and $B$ occur. Because of this, the intersection of events can be thought of in terms of the **AND** operator. Similarly, the probability of $A \cap B$ is restricted by the probabilities of $A$ and $B$. Specifically,

$$P(A \cap B) \leq P(A) \quad \text{and} \quad P(A \cap B) \leq P(B)$$

**Example 2**

Suppose we roll two 6-sided dice, with event $A$ being that the values of both dice are even and event $B$ being that the sum of the values of both dice is greater than 6. What is $A \cap B$?

**Solution 2**

In this case, $A \cap B$ is the event that the values of both dice are even AND the sum of the values of both dice are greater than 6.

Additionally, there is no limit to how many events we can have in an intersection. All of the following intersections are completely valid:

$$A \cap B \cap C$$
$$A \cap B \cap C \cap D$$
$$A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6 \cap \ldots \text{ (infinitely many events)}$$

**Activity 2**

Suppose we have a standard deck of 52 cards and we randomly draw a card. Determine the probability of the following intersection of events. Show your work.

(a) $A \cap B =$ the card is a **Spade** AND the card is a **7**.

(b) $A \cap B =$ the card is a **Spade** AND the card is a **Diamond**.

(c) $A \cap B =$ the card is greater than **5** AND the card is not a face-card.

## Independent and Dependent Events

**Independent events** are events that do not affect one another, meaning that the results do not depend on each other. For example, the outcomes of multiple coin flips are independent events because the result of each coin flip does not depend on any of the other results. This also means that the probability of independent events are not affected by each other, giving us the formulas:

$$P(A \cap B) = P(A) \times P(B) \quad \text{or} \quad P(A) = \frac{P(A \cap B)}{P(B)} \quad \text{or} \quad P(B) = \frac{P(A \cap B)}{P(A)}$$

when $A$ and $B$ are independent events.

**Dependent events** are events that do affect one another, meaning that the results (and probabilities) are influenced by previous results. For example, if we are drawing from a bag of different coloured marbles and we pick a red marble the first time, and then go to draw a second marble without putting the red marble back in the bag (without replacement), then we have dependent events. This is because there is one fewer red marble in the bag than before, meaning the probability for each marble getting picked has changed.

In Activity 2, part c) is an example of dependent events because either of the events $A$ or $B$ being true will affect the other. If the card is greater than 5 then there is a lower probability that the card is not a face-card, and and if the card is not a face-card then there is a lower probability that the card is greater than 5.

**Activity 3**

Determine if the following events are independent or dependent. If they are independent then solve for the probability of the intersection, $A \cap B$.

(a) When flipping a coin, $A =$ first flip is Heads, and $B =$ second flip is Tails.

(b) When flipping a coin, $A =$ first flip is Heads, and $B =$ second flip is Heads.

(c) When rolling a 6-sided die, $A =$ value is even, and $B =$ value is 2.

(d) In a student council election, there are 10 candidates, with 4 of them in grade 7 and 6 of them in grade 8. There are two positions available: President and Treasurer; with $A =$ the President is in grade 7, and $B =$ the Treasurer is in grade 8.

## Conditional Probability

**Example 3**

Let's go back to our standard deck of 52 cards. In Example 1, we calculated the probability of randomly drawing a Club from the deck to be 0.25. Suppose we randomly draw a card from the deck but this time we know the card is black. What is the probability of the card being a Club now?

**Solution 3**

Similar to Example 1, there are 13 possible Clubs, but since we know the card is black, there are now 26 possible cards that the randomly drawn card can be. So, the new $P(A)$ is $\frac{13}{26}$ which is 0.5.

This is the essence of conditional probability. As new information becomes available, the probability of an event gets updated given the information. We denote the probability of event $A$ given that $B$ has already occurred by $P(A \mid B)$.

We then get the following formulas for the probability of event $A$ given event $B$:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \text{or} \quad P(A \cap B) = P(A \mid B) \times P(B) \quad \text{or} \quad P(B) = \frac{P(A \cap B)}{P(A \mid B)}$$

We see that since the new probability of event $A$ is affected by the probability of event $B$, conditional probability deals with dependent events.

**Example 4**

We are drawing marbles from a sack that contains the following: 7 red marbles, 8 blue marbles, and 5 green marbles. We have the events: $A$ = the first marble is red, and $B$ = the second marble is blue.

(a) What is $P(B \mid A)$ if we are drawing without replacement? What is $P(A \cap B)$?

(b) What is $P(B \mid A)$ if we are drawing with replacement? What is $P(A \cap B)$?

**Solution 4**

(a) 'Without replacement' means that after we draw a marble, we don't place it back in the sack. Because of this, $A$ and $B$ are dependent. So, we initially have 20 marbles in the sack, and the probability of drawing a red marble first is $P(A) = \frac{7}{20} = 0.35$. Since we don't replace the red marble, the second draw has only 19 marbles in the sack. There are still 8 blue marbles in the sack, so $P(B \mid A) = \frac{8}{19}$. Using our formula above, we have that $P(A \cap B) = P(B \cap A) = P(B \mid A) \times P(A) = \frac{8}{19} \times \frac{7}{20} = \frac{56}{380} = \frac{14}{95}$.

(b) 'With replacement' means that after we draw a marble, we place it back in the sack. Because of this, $A$ and $B$ are independent. So, if we assume we draw a red marble on the first draw and then place it back in the sack, for the second draw we still have 20 marbles in the sack. There are still 8 blue marbles in the sack, so $P(B \mid A) = \frac{8}{20} = 0.4$. Using our formula above, we have that $P(A \cap B) = P(B \cap A) = P(B \mid A) \times P(A) = 0.4 \times 0.35 = 0.14$.

In part (b) of Example 4, we have that $A$ and $B$ are independent events and $P(B \mid A)$ is equal to the initial probability of drawing a blue marble. Thus, when $A$ and $B$ are independent, we have the following formula:

$$P(A \mid B) = P(A)$$

**Activity 4**

Find $P(A \mid B)$ for each of the following using the formulas on the previous page.

(a) $P(A \cap B) = 0.4$ and $P(B) = 0.6$

(b) $P(A \cap B) = 0$ and $P(B) = 0.7$

(c) $P(A \cap B) = 0.5$ and $P(B) = 1$

## Birthday Problem

**Example 5**

Suppose there is an unknown amount of people in a room and each person is asked for their birthday. Let $A =$ the event that at least 2 people have the same birthday. How many people would need to be in the room in order for $P(A)$ to be approximately 0.5?

**Solution 5**

Surprisingly, the answer is only **23** people.

Here is the link to the video that gives the full solution to the example above.

Birthday Problem Solution

Listed below are the formulas that were given the video:

**Number of Pairings:**

$$\frac{n \times (n-1)}{2}$$

**Probability That At Least 2 People Have The Same Birthday:**

$$P(A) = 1 - \left[\frac{365}{365} \times \frac{364}{365} \times \ldots \times \frac{365-n+1}{365}\right]$$

where $n$ is the number of people.