Grade 7/8 Math Circles
November 3, 2021
Polygonal Numbers - Solutions

Exercise Solutions

Exercise 1

Use systems of equation (method used in the previous lesson) to find the closed-form formula for the triangular numbers to double-check the formula we just derived.

Exercise 1 Solution

Substituting the first three terms into $T_n = an^2 + bn + c$ we get

$$1 = a + b + c \tag{1}$$
$$3 = 4a + 2b + c \tag{2}$$
$$6 = 9a + 3b + c \tag{3}$$

Solving the system above we get that $a = b = \frac{1}{2}$ and $c = 0$. Therefore the closed-form formula is

$$T_n = \frac{1}{2}n^2 + \frac{1}{2}n$$

Notice that although this formula looks different than the one derived above, they are equivalent. Expanding the first formula we have that

$$\begin{aligned} T_n &= \frac{1}{2}n(n+1) \\ &= \frac{1}{2}(n^2 + n) \\ &= \frac{1}{2}n^2 + \frac{1}{2}n \end{aligned}$$

which is equivalent to the formula we derived using systems of equations.

Exercise 2

How many handshakes will occur in a party of 35 guests?

Exercise 2 Solution

If there are 35 guests, then the number of handshakes would be $T_{35-1} = T_{34}$, the 34^{th} triangular number. Using the closed-form formula we derived, we have that

$$\begin{aligned} T_{34} &= \frac{34(34+1)}{2} \\ &= \frac{34(35)}{2} \\ &= \frac{1190}{2} \\ &= 595 \end{aligned}$$

So 595 handshakes need to occur for every guest to shake hands with everybody else.

Exercise 3

What is the 12^{th} term of the sequence of heptagonal numbers?

Exercise 3 Solution

Substituting $n = 12$ and $s = 7$ into the formula above we get

$$\begin{aligned} H_n &= \frac{1}{2}((7-2)12^2 - (7-4)12) \\ &= \frac{1}{2}((5)(144) - (3)(12)) \\ &= \frac{1}{2}(720 - 36) \\ &= \frac{1}{2}(684) \\ &= 342 \end{aligned}$$

Therefore the 12^{th} term of the sequence of heptagonal numbers is 342.

Problem Set Solutions

1. Recall that the n^{th} triangular number can be found using the formula $\dfrac{n^2+n}{2}$. Compute the 10^{th}, 17^{th}, and the 56^{th} triangular number.

Solution:

$$\begin{aligned} T_{10} &= \frac{(10)^2+10}{2} \\ &= \frac{110}{2} \\ &= 55 \end{aligned}$$

$$\begin{aligned} T_{17} &= \frac{(17)^2+17}{2} \\ &= \frac{289+17}{2} \\ &= \frac{306}{2} \\ &= 153 \end{aligned}$$

$$\begin{aligned} T_{56} &= \frac{(56)^2+56}{2} \\ &= \frac{3136+56}{2} \\ &= \frac{3192}{2} \\ &= 1596 \end{aligned}$$

2. Compute the following.

   (a) The 6^{th} term of the sequence of hexagonal numbers.

   (b) The 9^{th} term of the sequence of octagonal numbers.

   (c) The 5^{th} term of the sequence of icosagonal numbers (20 sides).

Solution:

(a) Using the closed-form formula for the sequence of hexagonal numbers given in the lesson and substituting in $n = 6$ we get

$$\begin{aligned} H_6 &= \frac{1}{2}(4(6)^2 - 2(6)) \\ &= \frac{1}{2}(4(36) - 12) \\ &= \frac{1}{2}(144 - 12) \\ &= \frac{1}{2}(132) \\ &= 66 \end{aligned}$$

(b) Using the closed-form formula for the sequence of octagonal numbers given in the lesson and substituting in $n = 9$ we get

$$\begin{aligned} O_6 &= \frac{1}{2}(6(9)^2 - 4(9)) \\ &= \frac{1}{2}(6(81) - 36) \\ &= \frac{1}{2}(486 - 36) \\ &= \frac{1}{2}(450) \\ &= 225 \end{aligned}$$

(c) We are not given the closed-form formula for the sequence of icosagonal numbers, but we can use the generalized formula with $s = 20$ and $n = 5$.

$$\begin{aligned} I_5 &= \frac{(20-2)(5^2) - (20-4)(5)}{2} \\ &= \frac{(18)(25) - (16)(5)}{2} \\ &= \frac{450 - 80}{2} \\ &= \frac{370}{2} \\ &= 185 \end{aligned}$$

3. Find a closed-form formula for the n^{th} term of the sequence of
   (a) Tridecagonal numbers (13 sides)
   (b) Enneadecagonal numbers (19 sides)
   (c) Icositetragonal numbers (24 sides)

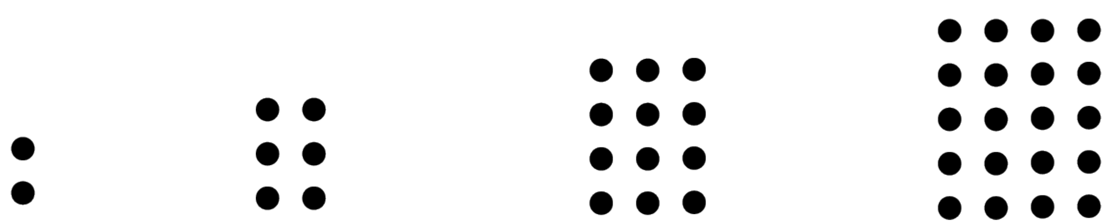
Solution: We can use the generalized formula and substitute in $s = 13$, $s = 19$, and $s = 24$ to obtain

(a) $\dfrac{(13-2)n^2-(13-4)n}{2} = \dfrac{11n^2-9n}{2}$

(b) $\dfrac{(19-2)n^2-(19-4)n}{2} = \dfrac{17n^2-15n}{2}$

(c) $\dfrac{(24-2)n^2-(24-4)n}{2} = \dfrac{22n^2-20n}{2} = 11n^2-10n$

4. An *oblong* number is the number of dots in a rectangular grid with one more row than column. The first four oblong numbers are 2, 6, 12, and 20, as shown below.



What is the 9^{th} oblong number?

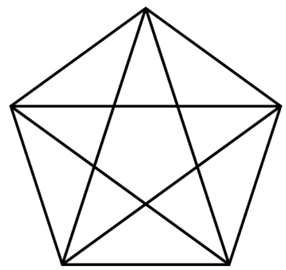
Solution: To calculate the number of dots in a rectangular grid, we use *width*$\times$*length*. Since the width of an oblong rectangle is equal to n and the length is equal to $n+1$, we can simply substitute $n = 9$ into the formula $n \times (n+1)$ to obtain the 9^{th} oblong number.

$$\begin{aligned} 9 \times (9+1) &= 9 \times 10 \\ &= 90 \end{aligned}$$

Therefore the 9^{th} oblong number is 90.

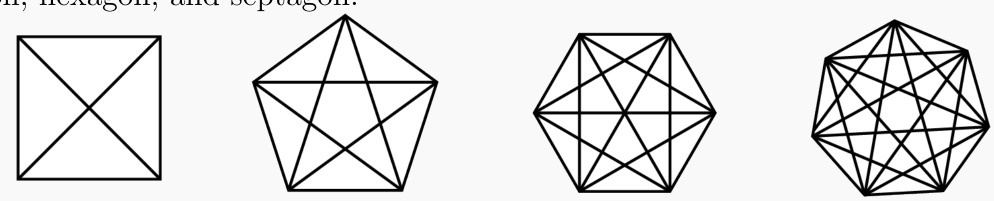
5. An n-gon is an n-sided regular polygon. For example, a 5-gon is a pentagon.

A diagonal of a polygon is a line segment drawn between any two vertices (corners). For example, a pentagon has 5 diagonals, shown on the image below.



How many diagonals does an n-gon have? *(Hint: try sketching out the first few n-gons starting from the triangle and counting the number of diagonals.)*

Solution: A triangle has no diagonals, so let us count the number of diagonals for a square, pentagon, hexagon, and septagon.



Counting the number of diagonals within each shape, we have that

Shape	**Number of Diagonals**
Triangle	0
Square	2
Pentagon	5
Hexagon	9
Septagon	14

Notice that these numbers are one less than the triangular numbers.
Note that we are starting from $s = 3$, but we should be starting from $n = 1$. Therefore, we must use $n = s - 2$ for our calculations.
Putting everything together, the number of diagonals that an n-gon has is the $(n-2)^{\text{th}}$ triangular number minus 1:

$$T_{n-2} - 1 = \frac{(n-2)(n-1)}{2} - 1$$

6. Compute the first 10 triangular numbers using the closed-form formula for the sequence of triangular numbers. Then, compute the first 5 hexagonal numbers by using the generalized closed-form formula. What do you notice about these two sequences?

Solution: Using the formula $T_n = \frac{n(n+1)}{2}$ and substituting $n = 1, ..., 10$ we get

$$\begin{aligned}
T_1 &= \frac{1(1+1)}{2} = 1 \\
T_2 &= \frac{2(2+1)}{2} = 3 \\
T_3 &= \frac{3(3+1)}{2} = 6 \\
T_4 &= \frac{4(4+1)}{2} = 10 \\
T_5 &= \frac{5(5+1)}{2} = 15 \\
T_6 &= \frac{6(6+1)}{2} = 21 \\
T_7 &= \frac{7(7+1)}{2} = 28 \\
T_8 &= \frac{8(8+1)}{2} = 36 \\
T_9 &= \frac{9(9+1)}{2} = 45 \\
T_{10} &= \frac{10(10+1)}{2} = 55
\end{aligned}$$

So we have that the first 10 terms of the sequence of triangular numbers is

$$\{1, 3, 6, 10, 15, 21, 28, 36, 45, 55\}$$

Using the formula $H_n = \frac{1}{2}(4n^2 - 2n) = 2n^2 - n$ (given in the lesson) and substituting

$n = 1, ..., 5$ we get

$$\begin{aligned}
H_1 &= 2(1)^2 - 1 = 1 \\
H_2 &= 2(2)^2 - 2 = 6 \\
H_3 &= 2(3)^2 - 3 = 15 \\
H_4 &= 2(4)^2 - 4 = 28 \\
H_5 &= 2(5)^2 - 5 = 45
\end{aligned}$$

So we have that the first 5 terms of the sequence of hexagonal numbers is

$$\{1, 6, 15, 28, 45\}$$

Notice that the hexagonal numbers are the odd triangular numbers! This is a unique relationship between hexagonal and triangular numbers.

7. Without using a calculator, find the sum of the integers from 1 to 100. In other words, find

$$1 + 2 + 3 + ... + 100$$

Use a similar strategy to find

$$18 + 19 + 20 + ... + 77 + 79$$

Solution: Notice that the n^{th} triangular number is equal to $T_n = 1 + 2 + 3 + ... + n$. Therefore, we can find $1 + 2 + 3 + ... + 100$ by finding T_{100}.

$$T_{100} = \frac{100(100 + 1)}{2} = \frac{10100}{2} = 5050$$

Gauss, a famous mathematician, discovered this formula as a child when his teacher told his class to find the sum of the numbers 1 through to 100 to keep them busy in class. The young Gauss found a pattern in the summation. If you add the first and last number, you obtain 101, if you add the second and second-to-last number, you also obtain 101. This pattern holds all the way down to the middle two numbers - 50 and 51. Therefore, all you have to do is multiply the sum of each pair (101) by the number of pairs (50) to obtain 5050 as the final sum. We can use a similar strategy for the second sum.

Notice that $18+79 = 97$, $19+78 = 97$, and so on. If we continue matching these pairs, we will end up at the last pair, $48 + 49 = 97$. We can find the number of pairs by computing

$$\frac{79 - 18 + 1}{2}$$

where we divide the number of elements in our sum by 2. Putting everything together, we can evaluate the sum

$$18 + 19 + ... + 77 + 78 = \frac{(79 - 18 + 1)(97)}{2} = \frac{6014}{2} = 3007$$

8. Using square numbers, compute the sum of the first 15 consecutive positive even integers. Then, find a general closed-form formula for the sum of the first n consecutive positive even integers.

Solution: We know that we can use square numbers to find the sum of consecutive positive even integers. Therefore, we only need to make a small modification to compute the sum of consecutive positive even integers.

Notice that every time we add the next consecutive even integer, we are essentially adding (the next consecutive odd integer + 1). Therefore, if we want to add the first n consecutive positive even integers, we just need to compute the sum of the first n consecutive positive odd integers, then add n to that sum.

So, the sum of the first 15 consecutive positive even integers is "the sum of the first 15 consecutive positive odd integers +15", and using the square numbers, is equal to

$$(15)^2 + 15 = 240$$

So the general formula for the sum of the first n consecutive positive even integers is

$$n^2 + n$$