Grade 7/8 Math Circles
November 17
Different Base Counting Systems

Decimal Counting System

The word "base" has different meanings in various contexts. In the context of counting systems, a base is defined as the total count of digits used to express numbers.

The **decimal counting system** is the counting system we are all familiar with. It consists of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, which we can use to measure different quantities. In other words, it is a counting system with base 10. Note that in this context, decimal does not refer to a decimal point in a number like 3.14, but rather the entire counting system. Let us look at how we use these 10 digits to represent an arbitrarily large quantity.

The digits themselves can count up to 9, but what happens after that? We use "10" to denote the next quantity, but what are we really doing? We are putting 1 in the "tens" place value and 0 in the "ones" place value and continue to count up using the available digits. Once we get to 99, we put 1 in the "hundreds" place value and 0's in the "tens" and "ones" place values. This pattern continues until we reach the desired quantity we want.

When we see the number 1508, how do we interpret it using the decimal system? Let us break the number down into its digits.

$$\begin{aligned}1508 &= 1000 + 500 + 8\\ &= 1 \times 1000 + 5 \times 100 + 0 \times 10 + 8 \times 1\\ &= 1 \times 10^3 + 5 \times 10^2 + 0 \times 10^1 + 8 \times 10^0\end{aligned}$$

When we break the number down, we see that we are essentially multiplying each digit by a power of 10.

(A note on powers: given two integers a and b, we say that a^b is "a to the power of b", which is equivalent to "a multiplied with iteslf b times". For example, $10^3 = 10 \times 10 \times 10 = 1000$. By definition, any number to the power of 0 is equal to 1. We say that a is the "base" and b is the "exponent". This is also why we refer to this counting system as "base 10".)

Throughout this lesson, we will be using notation to differentiate between the various counting systems when it is not clear from context what the base is. We will use the subscript "10" to denote a number in the decimal counting system, the subscript "2" to denote a number in the binary counting system, and the subscript "16" to denote a number in the hexadecimal counting system. Some examples could be 156_{10}, 1101010_2, and $6FA4_{16}$. This notation is necessary because numbers like 101_2, 101_{10}, and 101_{16} all represent a different quantity in their respective counting systems (refer to Exercise 5).

Binary Counting System

The **binary counting system** is the base 2 counting system, meaning that the only digits used to represent numbers are 0 and 1. The way we count in binary is the same as how we count in decimal. When we run out of digits to use, we set the value of the current place value back to 0 and add 1 to the next place value. The first few binary numbers are

$$0, 1, 10, 11, 100, 101, 110, 111, 1000, ...$$

and so on.

Let us explore how we represent decimal numbers using the binary counting system.

Binary	Expansion	Decimal
0	0×2^0	0
1	1×2^0	1
10	$1 \times 2^1 + 0 \times 2^0$	2
11	$1 \times 2^1 + 1 \times 2^0$	3
100	$1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$	4
101	$1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$	5
110	$1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$	6
111	$1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$	7
1000	$1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$	8

The "Expansion" column breaks down the binary number into its digits multiplied by a power of 2 (recall how we broke down the decimal number 1508 on the first page). If you evaluate each expansion, you will obtain the decimal value of that binary number, shown in the third column. From the table, we can see that the binary system works exactly like the decimal system, with the only difference being the number of available digits and the base.

Converting Between Binary and Decimal

Let's say we want to convert 1100101_2 to a decimal number. We can use a conversion chart like so:

Digit	1	1	0	0	1	0	1
Exponent Value	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Numerical Value	64	32	16	8	4	2	1

Note that you only need as many columns as there are digits in the binary number. In this example, since 1100101 is a 7-digit binary number, the chart needs 7 columns (and a column for the labels).

Once you set up the chart, simply multiply the digit by the numerical value for each column, and add the columns together. In this example, we have that

$$\begin{aligned} 1100101 &= 1 \times 64 + 1 \times 32 + 1 \times 4 + 1 \times 1 \\ &= 64 + 32 + 4 + 1 \\ &= 101 \end{aligned}$$

Exercise 1

Convert 11010011_2 to a decimal number.

Now let us convert some decimal numbers to binary numbers. Suppose we want to convert 72_{10} into a binary number. We would like to find the greatest power of 2 that is less than or equal to 72, subtract that value from 72, and then continue the algorithm with the difference. You can use a similar chart as above to convert any decimal number less than or equal to 1024 into a binary number. If you need to convert a binary number with more digits, simply expand the chart to include more columns.

Binary Digit											
Powers of 2	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Numerical Value	1024	512	256	128	64	32	16	8	4	2	1

We can see that the greatest power of 2 less than or equal to 72 is $2^6 = 64$. Therefore, we can put a 1 in that column.

Binary Digit					1						
Powers of 2	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Numerical Value	1024	512	256	128	64	32	16	8	4	2	1

Then, let us subtract 64 from 72. We get that $72 - 64 = 8$. The greatest power of 2 less than or equal to 8 is $2^3 = 8$, so we can put a 1 in that column.

Binary Digit					1			1			
Powers of 2	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Numerical Value	1024	512	256	128	64	32	16	8	4	2	1

Since $8 - 8 = 0$, we have successfully written 72 as $2^6 + 2^3$. Therefore, we can fill in the rest of the binary digits with 0's.

Binary Digit	0	0	0	0	1	0	0	1	0	0	0
Powers of 2	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Numerical Value	1024	512	256	128	64	32	16	8	4	2	1

Like with the decimal system, the digits are written, left to right, from the highest power of 2 to the lowest power of 2. In all counting systems, we ignore leading 0's. Therefore, 72_{10} is equal to 1001000_2.

Exercise 2

Convert 851_{10} to a binary number.

The binary counting system is used by all computers and many electronical devices. The circuits in a computer's processor are made up of billions of transistors, which are tiny switches activated by electronic signals (positive versus negative charge). These switches only have two possible positions (on and off), so the digits 1 and 0 reflect whether a transistor is "on" or "off". Since it is easy for computers to process switches (similar to answering a "yes" or "no" question) rather than to process a spectrum of possible choices, all the information processed by a computer is stored using the binary system. These two binary digits are also the smallest unit of data in computing, known as a "bit".

Hexadecimal Counting System

The **hexadecimal counting system** (or simply "hex") uses base 16. The 16 digits used by the hexadecimal systems are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, and F.

Conversion between decimal and hexadecimal requires associating each hex digit with its equivalent value in decimal.

Hex Digit	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Decimal Value	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Using a similar table as the one we used for binary, we can multiply each hex digit with a power of 16 to convert a hexadecimal number to a decimal number.

For example, if we wanted to convert B7E4_{16} into a decimal number, we can use the table

Hex Digit	B	7	E	4
Exponent Value	16^3	16^2	16^1	16^0
Numerical Value	4096	256	16	1

and obtain that

$$\begin{aligned} B7E4 &= B \times 4096 + 7 \times 256 + E \times 16 + 4 \times 1 \\ &= 11 \times 4096 + 7 \times 256 + 14 \times 16 + 4 \times 1 \\ &= 47076 \end{aligned}$$

Now let us convert decimal numbers to hexadecimal. You can do this in the same way as the method for binary to decimal (instead of finding the highest power of 2 less than or equal to the number, use the highest power of 16), but we will demonstrate a more efficient algorithm, which converts decimal numbers into a number in another base (16 in this case).

Suppose we want to convert 7562_{10} into a hexadecimal number. We will first use long division to divide 7562 by 16 and write down the quotient and remainder. We will use the table below to help us.

Division by 16	**Quotient**	**Remainder (Decimal)**	**Remainder (Hex)**
$7562 \div 16$	472	10	

To fill in the rest of the rows, we will divide the previous row's quotient by 16 until we reach a quotient of 0.

Division by 16	Quotient	Remainder (Decimal)	Remainder (Hex)
$7562 \div 16$	472	10	
$472 \div 16$	29	8	
$29 \div 16$	1	13	
$1 \div 16$	0	1	

Next, we will convert the remainder from a decimal number to a hex digit.

Division by 16	Quotient	Remainder (Decimal)	Remainder (Hex)
$7562 \div 16$	472	10	A
$472 \div 16$	29	8	8
$29 \div 16$	1	13	D
$1 \div 16$	0	1	1

Finally, write down the remainder column in hex from bottom to top. In this case, 7562_{10} is equal to 1D8A in hexadecimal.

Stop and Think

Why does this algorithm work? Can you understand why we perform each step of the algorithm?

Exercise 3

Convert A57BF_{16} to decimal. Then, convert 28660_{10} to hexadecimal.

Converting Between Binary and Hexadecimal

The hexadecimal counting system is primarily used in computing to express binary values in a more readable way. Each hexadecimal digit is used to represent a 4-digit binary number, where 0 in hexadecimal is 0000 in binary and F in hexadecimal is 1111 in binary. This makes it easier for humans to interpret long binary strings of 1's and 0's that are used by computers. Since a binary digit is a "bit", each hexadecimal digit takes up 4 bits of storage, equivalent to half a byte (also known as a "nibble").

Converting between binary and hexadecimal is quite easy. All we need is this conversion table.

Hex	0	1	2	3	4	5	6	7
Binary	0000	0001	0010	0011	0100	0101	0110	0111
Hex	8	9	A	B	C	D	E	F
Binary	1000	1001	1010	1011	1100	1101	1110	1111

When we want to convert between the two counting systems, simply use this table and remove any leading zeros. For example, 10101011110011011110 in binary is ABCDE in hexadecimal. Similarly, 1357_{16} is equal to 1001101010111_2. From this example alone, we can see why humans prefer to use hexadecimal rather than binary.

Exercise 4

Convert $F14C27_{16}$ to a binary number.

Exercise 5

Convert 101_2, 101_{10}, and 101_{16} into decimal values.

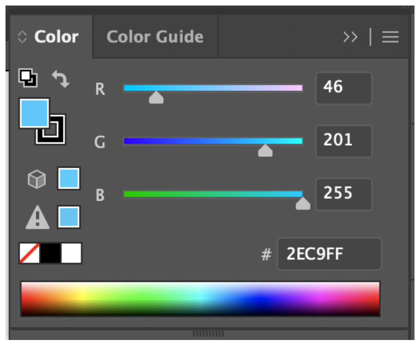
Hexadecimal Colour Spectrum

Hex is used in many areas of computing, such as the memory locations of error messages and the digital colour spectrum. Knowing how to interpret and use hex numbers is incredibly important for web design, such as HTML and CSS. Let us take a look at how hexadecimal numbers are used to represent digital colours.

If you have ever used an imaging editing program like Photoshop, you may have noticed that each colour is identified and stored using a hex number, in the format "#RRGGBB" (R for reds, G for greens, and B for blues), such as "#251CE4". The "#" symbol indicates that this number is in the hexadecimal counting system. When you combine some amount of red, green, and blue light and mix them together, you will obtain a unique colour. For example, #000000 is the colour black (no reds, blues, nor greens will create the darkest shade), and #FFFFFF is the colour white (the strongest lights combined together will create the brightest shade).

Recall that each hex digit takes up 4 bits of memory, and so 2 hex digits take up 1 byte (8 bits) of memory. The possible values for each colour goes from 00_{16} (0_{10}) to FF_{16} (255_{10}), so in total there are 256 possible values for each of the 3 colours. This gives us a total spectrum of 256 reds $\times$ 256 blues × 256 greens, which can display more than 16 million colours! This is referred to the **true color spectrum**.

Here is a screenshot of the colour picker in Adobe Illustrator. You can see that we picked a sky blue colour. You can also see the amount of reds, greens, and blues used by the computer to create this colour, and the unique hex code that represents this colour.



Exercise 6

Verify that $46_{10} = 2E_{16}$, $201_{10} = C9_{16}$, and $255_{10} = FF_{16}$.

Stop and Think

What is the hex code used to represent pure red? What about pure green or pure blue?

Other Counting Systems

Other than the three counting systems we focused on today, there are many other counting systems with different bases. A counting system with base B will have digits 0 to $B-1$, and the place values are exponential values with B as the base. For example, a “base 6” counting system would have 0, 1, 2, 3, 4, and 5 as digits.

Stop and Think

Are there restrictions on what number B can be?