Grade 7/8 Math Circles

November 17, 2021

Counting Systems - Solutions

Exercise Solutions

Exercise 1

Convert 11010011_2 to a decimal number.

Exercise 1 Solution

Binary Digit	1	1	0	1	0	0	1	1
Exponent Value	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Numerical Value	128	64	32	16	8	4	2	1

$$\begin{aligned} 11010011 &= 1 + 1 \times 64 + 1 \times 16 + 1 \times 2 + 1 \times 1 \\ &= 128 + 64 + 16 + 2 + 1 \\ &= 211 \end{aligned}$$

Exercise 2

Convert 851_{10} to a binary number.

Exercise 2 Solution

The greatest power of 2 less than or equal to 851 is $2^9 = 512$.

Binary Digit		1									
Powers of 2	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Numerical Value	1024	512	256	128	64	32	16	8	4	2	1

Since $851 - 512 = 339$, and the greatest power of 2 less than or equal to 339 is $2^8 = 256$.

Binary Digit		1	1								
Powers of 2	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Numerical Value	1024	512	256	128	64	32	16	8	4	2	1

Since $339 - 256 = 82$, and the greatest power of 2 less than or equal to 83 is $2^6 = 64$.

Binary Digit		1	1		1						
Powers of 2	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Numerical Value	1024	512	256	128	64	32	16	8	4	2	1

Since $83 - 64 = 19$, and the greatest power of 2 less than or equal to 19 is $2^4 = 16$.

Binary Digit		1	1		1		1				
Powers of 2	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Numerical Value	1024	512	256	128	64	32	16	8	4	2	1

Since $19 - 16 = 3$, and the greatest power of 2 less than or equal to 3 is $2^1 = 2$.

Binary Digit		1	1		1		1			1	
Powers of 2	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Numerical Value	1024	512	256	128	64	32	16	8	4	2	1

Finally, $3 - 2 = 1$, and the greatest power of 2 less than or equal to 1 is $2^0 = 1$.

Binary Digit		1	1		1		1			1	1
Powers of 2	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Numerical Value	1024	512	256	128	64	32	16	8	4	2	1

Since $1 - 1 = 0$, we can we fill the rest of the columns with 0's.

Binary Digit	0	1	1	0	1	0	1	0	0	1	1
Powers of 2	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Numerical Value	1024	512	256	128	64	32	16	8	4	2	1

After omitting the leading 0's, we get that 851_{10} is equal to 1101010011 in binary.

Exercise 3

Convert $A57BF_{16}$ to decimal. Then, convert 28660_{10} to hexadecimal.

Exercise 3 Solution

Part 1: Using the chart below,

Digit	A	5	7	B	F
Decimal Digit	10	5	7	11	15
Exponent Value	16^4	16^3	16^2	16^1	16^0
Numerical Value	65536	4096	256	16	1

we have that

$$\begin{aligned} A57BF &= 10 \times 65536 + 5 \times 4096 + 7 \times 256 + 11 \times 16 + 15 \times 1 \\ &= 655360 + 20480 + 1792 + 176 + 15 \\ &= 677823 \end{aligned}$$

Part 2: We fill in the values in the chart below and write down the hex remainder column from bottom to top.

Division by 16	**Quotient**	**Remainder (Decimal)**	**Remainder (Hex)**
$28660 \div 16$	1791	4	4
$1791 \div 16$	111	15	F
$111 \div 16$	6	15	F
$6 \div 16$	0	6	6

Therefore 28660_{10} is equal to 6FF4 in hexadecimal.

Exercise 4

Convert $F14C27_{16}$ to a binary number.

Exercise 4 Solution

Using the conversion table, $F14C27_{16}$ is 1111 0001 0100 1100 0010 0111 in binary (broken into groups of 4 for readability).

Exercise 5

Convert 101_2, 101_{10}, and 101_{16} into decimal values.

Exercise 5 Solution

101_{10} is already in the decimal counting system.

Using the binary to decimal conversion chart:

Binary Digit	1	0	1
Powers of 2	2^2	2^1	2^0
Numerical Value	4	2	1

we have that $101_2 = 1 \times 4 + 1 \times 1 = 5$ in decimal.

Using the hexadecimal to decimal conversion chart:

Hex Digit	1	0	1
Decimal Value	1	0	1
Exponent Value	16^2	16^1	16^0
Numerical Value	256	16	1

we have that $101_{16} = 1 \times 256 + 0 \times 16 + 1 \times 1 = 257$ in decimal.

Exercise 6

Verify that $46_{10} = 2E_{16}$, $201_{10} = C9_{16}$, and $255_{10} = FF_{16}$.

Exercise 6 Solution

We can either convert the values from decimal to hex, or from hex to decimal, to show that they are equivalent. We will convert each hex number to decimal in this exercise solution.

$2E_{16} = 2 \times 16^1 + E \times 16^0 = 2 \times 16 + 14 \times 1 = 32 + 14 = 46$ in decimal.

$C9_{16}$ = C$\times 16^1 + 9 \times 16^0 = 12 \times 16 + 9 \times 1 = 192 + 9 = 201$ in decimal.
FF_{16} = F$\times 16^1$+ F $\times 16^0 = 15 \times 16 + 15 \times 1 = 240 + 15 = 255$ in decimal.

Problem Set Solutions

1. Convert the following numbers into the decimal counting system.

(a) 1001001_2 (b) 101001011_2 (c) 11110101_2 (d) $71FA_{16}$ (e) ACE_{16}

Solution:

(a) $1001001_2 = 1 \times 2^6 + 1 \times 2^3 + 1 \times 2^0 = 73_{10}$

(b) $101001011_2 = 1 \times 2^8 + 1 \times 2^6 + 1 \times 2^3 + 1 \times 2^1 + 1 \times 2^0 = 331_{10}$

(c) $11110101_2 = 1 \times 2^7 + 1 \times 2^6 + 1 \times 2^5 + 1 \times 2^4 + 1 \times 2^2 + 1 \times 2^0 = 245_{10}$

(d) $71FA_{16} = 7 \times 16^3 + 1 \times 16^2 + F \times 16^1 + A + 16^0 = 28672 + 256 + 15 \times 16 + 10 \times 1 = 29178_{10}$

(e) $ACE_{16} = A \times 16^2 + C \times 16^1 + E \times 16^0 = 10 \times 16^2 + 12 \times 16^1 + 14 \times 1 = 2766_{10}$

2. Convert the following numbers to into the binary counting system.

(a) 47_{10} (b) 156_{10} (c) 222_{10} (d) $71C3_{16}$

Solution:

(a) $47_{10} = 1 \times 2^5 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 101111_2$

(b) $156_{10} = 1 \times 2^7 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 = 10011100_2$

(c) $222_{10} = 1 \times 2^7 + 1 \times 2^6 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 = 11011110_2$

(d) $71C3_{16} = 111000111000011_2$, using the conversion table on page 9 of the lesson pdf.

3. Convert the following numbers to into the hexadecimal counting system.

(a) 58_{10} (b) 7180_{10} (c) 101101010010_2 (d) 100000100_2

Solution:

(a) $58_{10} = 3 \times 16^1 + 10 \times 10^0 = 3A_{16}$

(b) $7180_{10} = 1 \times 16^3 + 12 \times 16^2 + 12 \times 16^0 = 1 \times 16^3 + C \times 16^2 + 0 \times 16^1 + C \times 16^0 = 1C0C_{16}$

(c) $101101010010_2 = B52_{16}$ using the conversion table on page 9 of the lesson pdf.

(d) $100000100_2 = 000100000100_2 = 104_{16}$ using the conversion table on page 9 of the lesson pdf. Here, we added leading 0's in order to make the total number of binary digits equal to a multiple of 4, so we can use the conversion table.

4. It is possible to perform addition in different base counting systems. In base 10, we perform addition by first adding the rightmost digit, and if the sum exceeds the number of available digits, we "carry over" 1 to the next highest place value. Addition works the same way in a different base, the only difference being the number of available digits there are to use.

Suppose we want to calculate $101 + 11$ in binary. We will start by adding the right most digit. $1 + 1 = 10$ in binary, so we carry over a 1 to the next place value, while the current place value becomes 0.

$$\begin{array}{r} \scriptstyle 1 \\ 101 \\ +11 \\ \hline 0 \end{array}$$

We carried a 1 to the second place value, so the next place value becomes $1 + 0 + 1 = 10$, and we carry over a 1 to the next place value and put a 0 in the current place value.

$$\begin{array}{r} \scriptstyle 1 \\ 101 \\ +11 \\ \hline 00 \end{array}$$

Finally, the next place value is $1 + 1 = 10$, and so the final answer is $101 + 11 = 1000$.

$$\begin{array}{r} 101 \\ +11 \\ \hline 1000 \end{array}$$

Perform the following calculations in binary. Think of what each place value represents and how to carry values over each place value (as with base 10 addition).

$$\text{(a)}\ \begin{array}{r}10\\+1\\\hline\end{array}\qquad\text{(b)}\ \begin{array}{r}1001\\+110\\\hline\end{array}\qquad\text{(c)}\ \begin{array}{r}10\\+10\\\hline\end{array}\qquad\text{(d)}\ \begin{array}{r}110\\+10\\\hline\end{array}\qquad\text{(e)}\ \begin{array}{r}111\\+11\\\hline\end{array}$$

> *Solution*:
>
> $$\text{(a)}\ \begin{array}{r}10\\+1\\\hline 11\end{array}\qquad\text{(b)}\ \begin{array}{r}1001\\+110\\\hline 1111\end{array}\qquad\text{(c)}\ \begin{array}{r}10\\+10\\\hline 100\end{array}\qquad\text{(d)}\ \begin{array}{r}110\\+10\\\hline 1000\end{array}\qquad\text{(e)}\ \begin{array}{r}111\\+11\\\hline 1010\end{array}$$

5. Similar to the question above, perform the following calculations in hexadecimal. Think of what each place value represents and how to carry values over each place value (as with base 10 addition).

$$\text{(a)}\ \begin{array}{r}17\\+6\\\hline\end{array}\qquad\text{(b)}\ \begin{array}{r}\text{AB}\\+\text{C}\\\hline\end{array}\qquad\text{(c)}\ \begin{array}{r}\text{71D}\\+\text{F}\\\hline\end{array}\qquad\text{(d)}\ \begin{array}{r}\text{A}\\+\text{B}\\\hline\end{array}\qquad\text{(e)}\ \begin{array}{r}\text{F}\\+1\\\hline\end{array}$$

You may refer to the hexadecimal to decimal conversion chart below.

Hex Digit	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Decimal Value	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

> *Solution*: Every time we obtain a digit that is greater than 15, we must carry over 1 to the next place value. The hexadecimal to decimal conversion chart above can be very useful for calculations.
>
> $$\text{(a)}\ \begin{array}{r}17\\+6\\\hline \text{1D}\end{array}\qquad\text{(b)}\ \begin{array}{r}\text{AB}\\+\text{C}\\\hline \text{B7}\end{array}\qquad\text{(c)}\ \begin{array}{r}\text{71D}\\+\text{F}\\\hline \text{72C}\end{array}\qquad\text{(d)}\ \begin{array}{r}\text{A}\\+\text{B}\\\hline 15\end{array}\qquad\text{(e)}\ \begin{array}{r}\text{F}\\+1\\\hline 10\end{array}$$

6. It is also to perform subtraction in different base counting systems. In a counting system with any base, we start by subtracting from the rightmost digit. If we obtain a negative difference, we can "borrow 1" from the next highest place value.

   Perform the following subtractions in binary and hexadecimal. Think of what each place value represents and how to borrow values from the next highest place value (as with base 10 addition).

(a) $\begin{array}{r} 1111 \\ -10 \\ \hline \end{array}$ (b) $\begin{array}{r} 1001 \\ -11 \\ \hline \end{array}$ (c) $\begin{array}{r} 17A \\ -5 \\ \hline \end{array}$ (d) $\begin{array}{r} 1D \\ -B \\ \hline \end{array}$ (e) $\begin{array}{r} A37D1 \\ E9C \\ \hline \end{array}$

Solution:

(a) $\begin{array}{r} 1111 \\ -10 \\ \hline 1101 \end{array}$ (b) $\begin{array}{r} 1001 \\ -11 \\ \hline 110 \end{array}$ (c) $\begin{array}{r} 17A \\ -5 \\ \hline 175 \end{array}$ (d) $\begin{array}{r} 1D \\ -B \\ \hline 12 \end{array}$ (e) $\begin{array}{r} A37D1 \\ E9C \\ \hline A2935 \end{array}$

7. Convert 152_{10} to a base-5 number. You may use the same method that we used for hexadecimal numbers in the lesson.

Solution:

Division by 5	**Quotient**	**Remainder (Decimal)**	**Remainder (Base** 5**)**
$152 \div 5$	30	2	2
$30 \div 5$	6	0	0
$6 \div 5$	1	1	1
$1 \div 5$	0	1	1

Therefore 152_{10} is equal to 1102_5.

8. Is it possible for a counting system to be base 0? What about base 1?

Solution: It is not possible for a counting system to be base 0 nor base 1. In order for a counting system to be base 0, it means you will need to use 0 digits to represent numbers, which is impossible. In addition, for a counting system to be base 1, the only available digit is 0. Since $(0 \times 1^0 + 0 \times 1^1 + 0 \times 1^2 + \dots)$ will always equal 0, a base 1 counting system is also not possible, since we cannot represent numbers in this way.

9. Here are some problems regarding 5 digit numbers. State all your answers in the decimal counting system.

   (a) How many different 5-digit binary numbers are there? Note that the leading digit must be a 1 since we truncate all leading 0's. *(Hint: how would you do this in base* 10*?)*

   (b) How many different 5-digit binary numbers are there that have 1 as their last digit?

(c) How many different 5-digit base B numbers are there for any given B? Again, the leading digit cannot be 0. You may assume that $2 \leq B \leq 10$.

Solution:

(a) We know the first digit must be a 1, and the other 4 digits can either be a 1 or a 0. Therefore, there are $2 \times 2 \times 2 \times 2 = 16$ different 5-digit binary numbers.

Alternate solution: the largest 5-digit binary number is 11111, and the largest 4-digit binary number is 1111. Therefore, there are $11111 - 1111$ different 5-digit binary numbers. We can perform subtraction like so:

$$\begin{array}{r} 11111 \\ -1111 \\ \hline 10000 \end{array}$$

Binary Digit	1	0	0	0	0
Exponent Value	2^4	2^3	2^2	2^1	2^0
Decimal Value	16	8	4	2	1

Since 10000 is $1 \times 2^4 = 16$ in the decimal system, so there are 16 different 5-digit binary numbers.

(b) It is helpful to realize that all binary numbers that have 1 as their last digit are odd, and all binary numbers that have 0 as their last digit are even. Since there are 16_{10} 5-digit binary numbers, half of them must be odd and the other half must be even. Therefore, there are 8 different 5-digit binary numbers that have 1 as their last digit.

(c) We know the first digit must be non-zero, so there are (B−1) choices for the first digit, and there are B choices for the other 4 digits. Therefore, there are $(B-1) \times B \times B \times B \times B = (B-1) \times B^4$ different 5-digit base B numbers.

Alternate solution: the largest 5-digit base B number is the the number where all five digits are $(B-1)$, and the largest 4-digit base B number is the the number where all four digits are $(B-1)$. If you subtract the 4-digit number from the 5-digit number, you will obtain a number with $(B-1)$ as the first digit (with the B^4 as the place value) and 0's in all the other digits. Evaluating this in base 10, we can see that there are $(B-1) \times B^4$ different 5-digit base B numbers.

Binary Digit	$(B-1)$	0	0	0	0
Exponent Value	B^4	B^3	B^2	B^1	B^0

10. Suppose there is a base-26 counting system where the digits are the English alphabet, in the order from a to z. The conversion chart is as follows.

Letter	a	b	c	d	e	f	g	h	i	j	k	l	m
Decimal Value	0	1	2	3	4	5	6	7	8	9	10	11	12
Letter	n	o	p	q	r	s	t	u	v	w	x	y	z
Decimal Value	13	14	15	16	17	18	19	20	21	22	23	24	25

Translate the following secret message from base-10 to base-26. The numbers are separated by spaces, and each number represents a "word".

7838998 381 9171, 126182 8 44932!

You can use the chart below to help you.

Exponent Value	26^4	26^3	26^2	26^1	26^0
Numerical Value	456976	17576	676	26	1

Hint: you can either use the chart above to find teh greatest power of 26 that is less than or equal to the decimal number and use the subtraction algorithm (pages 3 and 4 of the lesson pdf), or use the long division algorithm (page 7 and 8 of the lesson pdf) and convert the remainders to base-26, which is much faster to compute.

Note: these types of secret messages are known as **cryptography**, where mathematical concepts or rules are used to encrypt messages into forms that are hard to decipher for anyone who is unaware of the rules. In this case, the rules refer to the base-26 counting system, as introduced above.

Solution: There are two ways to convert the numbers in base-26 to decimal. You can either use the chart above to find the greatest power of 26 that is less than or equal to the number and use the subtraction algorithm (pages 3 and 4 of the lesson pdf), or use the long division algorithm (page 7 and 8 of the lesson pdf) and convert the remainders to base-26. We will demonstrate the latter method here.

Division by 26	**Quotient**	**Remainder (Decimal)**	**Remainder (Letter)**
$7838998 \div 26$	301499	24	y
$301499 \div 26$	11596	3	d
$11596 \div 26$	446	0	a
$446 \div 26$	17	4	e
$17 \div 26$	0	17	r

Reading from bottom to top, the first word is "ready".

Division by 26	**Quotient**	**Remainder (Decimal)**	**Remainder (Letter)**
$381 \div 26$	14	17	r
$14 \div 26$	0	14	o

Reading from bottom to top, the second word is "or".

Division by 26	**Quotient**	**Remainder (Decimal)**	**Remainder (Letter)**
$9171 \div 26$	352	19	t
$352 \div 26$	13	14	o
$13 \div 26$	0	13	n

Reading from bottom to top, the third word is "not".

Division by 26	**Quotient**	**Remainder (Decimal)**	**Remainder (Letter)**
$126182 \div 26$	4853	4	e
$4853 \div 26$	186	17	r
$186 \div 26$	7	4	e
$7 \div 26$	0	7	h

Reading from bottom to top, the fourth word is "here".

8 in decimal is i in base-26, so the fifth word is i.

Division by 26	**Quotient**	**Remainder (Decimal)**	**Remainder (Letter)**
$44932 \div 26$	1728	4	e
$1728 \div 26$	66	12	m
$66 \div 26$	2	14	o
$2 \div 26$	0	2	c

Reading from bottom to top, the final word is "come".

Putting everything together and capitalizing the pronoun, the secret message is:

Ready or not, here I come!