Intermediate Math Circles

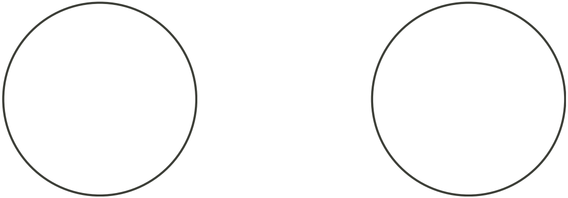
Rob Gleeson
Geometry II: Circles

rob.gleeson@uwaterloo.ca

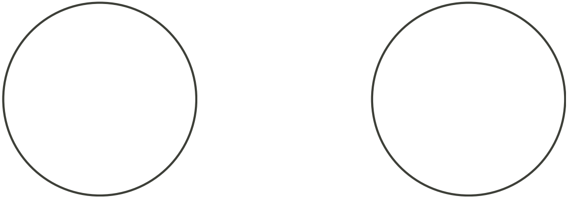
cemc.uwaterloo.ca

November 3 2021

What do we know about circles?

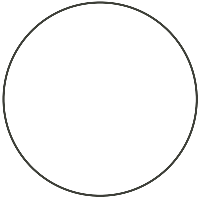


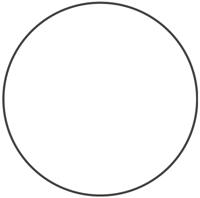
What do we know about circles?



- Circles are round.

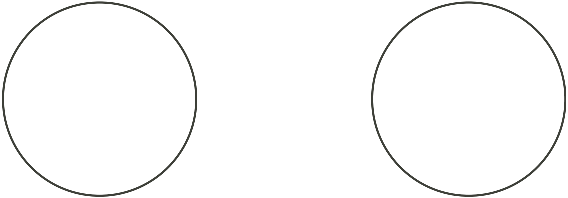
What do we know about circles?





- Circles are round.
- Diameter $= 2 \times$ radius

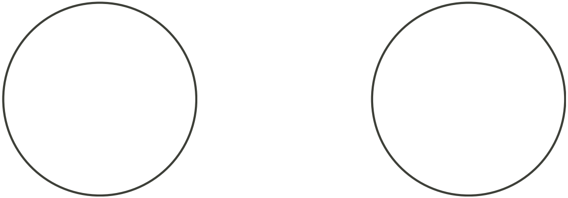
What do we know about circles?



- Circles are round.
- Diameter $= 2 \times$ radius
- $A = \pi r^2$

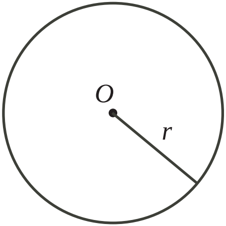
Circle Geometry

What do we know about circles?

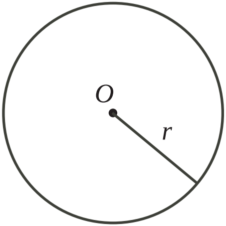


- Circles are round.
- Diameter $= 2 \times$ radius
- $A = \pi r^2$
- $C = \pi d$ or $C = 2\pi r$

Definition of a Circle



Definition of a Circle



A *circle* is a set of points in 2-space that are all equidistant from a fixed point. The fixed distance is called the *radius* and the fixed point is called the *centre*.

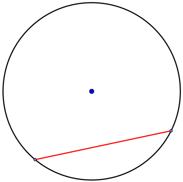
Definition of a Chord

Definition of a Chord

A *chord* is a line segment with its endpoints on the circumference of a circle.

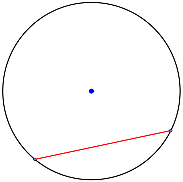
Definition of a Chord

A *chord* is a line segment with its end-points on the circumference of a circle.



Definition of a Chord

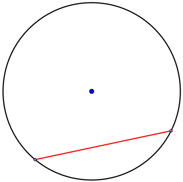
A *chord* is a line segment with its end-points on the circumference of a circle.



Definition of a Diameter

Definition of a Chord

A *chord* is a line segment with its end-points on the circumference of a circle.

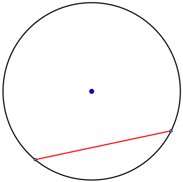


Definition of a Diameter

A *diameter* is a chord that passes through the centre of a circle.

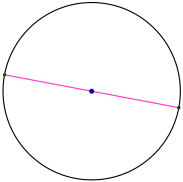
Definition of a Chord

A *chord* is a line segment with its end-points on the circumference of a circle.



Definition of a Diameter

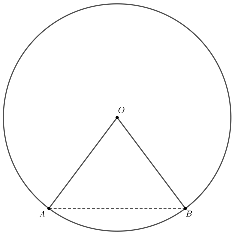
A *diameter* is a chord that passes through the centre of a circle.



Circle Theorems

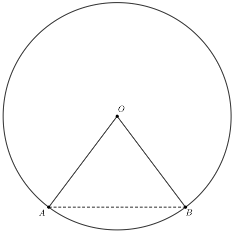
We are going to take a look at a number of theorems related to circles.
We will give some more definitions, then introduce some of the theorems.

A *central angle* is an angle whose vertex is at the centre that is subtended by a chord (or an arc) of a circle. In the diagram, O is the centre of the circle and therefore, $\angle AOB$ is a central angle.

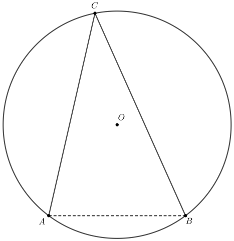


Central and Inscribed Angles

A *central angle* is an angle whose vertex is at the centre that is subtended by a chord (or an arc) of a circle. In the diagram, O is the centre of the circle and therefore, $\angle AOB$ is a central angle.

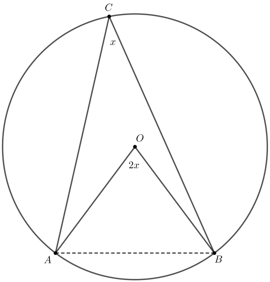


An *inscribed angle* is an angle whose vertex is on the circle that is subtended by a chord (or an arc) of a circle. In the diagram, $\angle ACB$ is a central angle.



Circle Theorems

Circle Theorem 1: The central angle subtended by a chord is twice the angle of an inscribed angle subtended by the same chord.

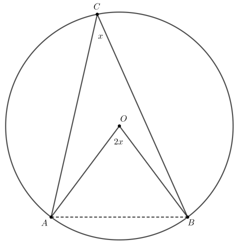


Proof of Circle Theorem 1.

There are two cases we need to look at:

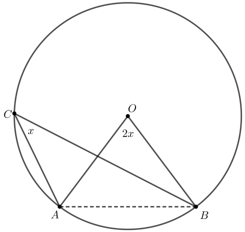
Case 1: The centre of the circle is in the inscribed angle.

We will prove this case over the next few pages.



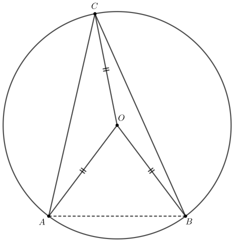
Case 2: The centre of the circle is outside the inscribed angle.

The proof will be asked as a question in the problem set.



Proof of Circle Theorem 1.

Case 1: The centre of the circle is in the inscribed angle.



Join C to O.

Therefore, OA = OC = OB since all three are radii of the same circle.

Now $\triangle AOC$ is isosceles and from the Isosceles Triangle Theorem.

Therefore, for some real number a, $\angle OCA = \angle OAC = a$. Therefore, $\angle COA = 180 - 2a$.

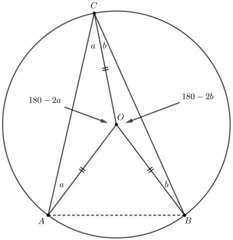
Similarly, we can show that for some real number b, $\angle OCB = \angle OBC = b$ and $\angle COA = 180 - 2b$.

(We will continue onto the next page.)

Circle Theorems

Now $\angle AOC, \angle BOC$, and $\angle AOB$ form a full rotation. Therefore,

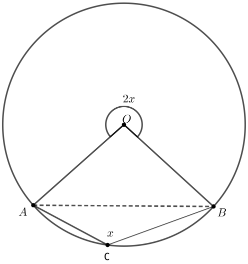
$$\begin{aligned}\angle AOC + \angle BOC + \angle AOB &= 360\\(180 - 2a) + (180 - 2b) + \angle AOB &= 360\\360 - 2a - 2b + \angle AOB &= 360\\\angle AOB &= 2a + 2b\\\angle AOB &= 2(a + b)\end{aligned}$$



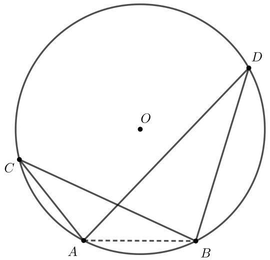
Now $\angle ACB = \angle ACO + \angle BCO = a + b$.
Therefore, $\angle AOB = 2\angle ACB$.
Therefore, the central angle subtended by a chord is twice the angle of an inscribed angle subtended by the same chord.

Circle Theorems

Note that the Circle Theorem 1 also works if the inscribed angle is obtuse.



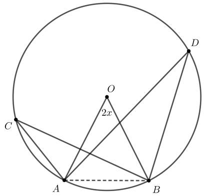
Circle Theorem 2: Two inscribed angles subtended by the same chord and on the same side of the chord are equal. This means for the following diagram $\angle ACB = \angle ADB$.



We will prove this theorem on the next page.

Proof of Circle Theorem 2.

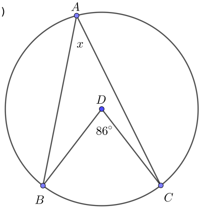
We will draw central angle subtended from chord AB. We will let $\angle AOB = 2x$.



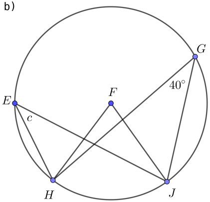
Now, we know $\angle ACB$ is an inscribed angle subtended from the chord AB and $\angle AOB$ is the central angle subtended from chord AB.
From Circle Theorem 1, $\angle ACB = \frac{1}{2}\angle AOB = \frac{1}{2}(2x) = x$.
Similarly, we can show that $\angle ADB = x$.
Therefore, $\angle ACB = \angle ADB = x$.
Therefore, two inscribed angles subtended by the same chord are equal.

For each question, find the value of the unknowns. Justify your answers.

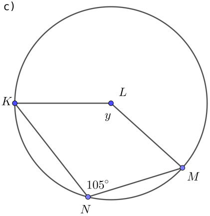
a)



b)



c)



Solutions are given on the next page.

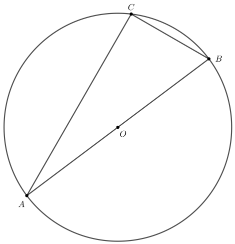
Circle Theorems Exercises Solutions

a) Since BC is a chord, $\angle BAC$ is an inscribed angle and $\angle BDC$ is a central angle. By Circle Theorem 1, $\angle BAC = \frac{1}{2}\angle BDC = 43$. Therefore $x = 43^\circ$.

b) Since HJ is a chord, $\angle HEJ$ and $\angle HGJ$ are inscribed angles. By Circle Theorem 2, $\angle HEJ = \angle HGJ = 40$. Therefore, $c = 40^\circ$.

c) Since KM is a chord, $\angle KNM$ is an inscribed angle and reflex angle KLM is the associated central angle. By Circle Theorem 1, $\angle KLM = 2\angle KNM = 210$. Now $210 + y = 360$ or $y = 150$.

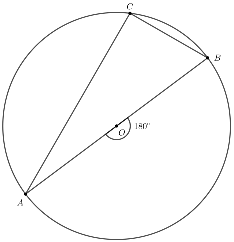
Circle Theorem 3: An inscribed angle subtended by a diameter is a right angle. In the diagram AB is a diameter and, therefore, $\angle ACB = 90^\circ$.



We will prove this on the next page.

Proof of Circle Theorem 3:

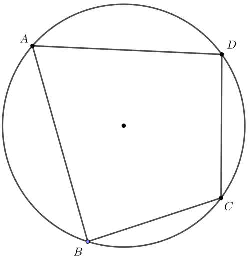
Central $\angle AOB = 180^\circ$ is subtended by AB.
$\angle ACB$ is an inscribed angle subtended by AB.
By Circle Theorem 1,
$\angle ACB = \frac{1}{2}\angle AOB = \frac{1}{2}(180^\circ) = 90^\circ$.



Therefore, an inscribed angle subtended by a diameter is a right angle.

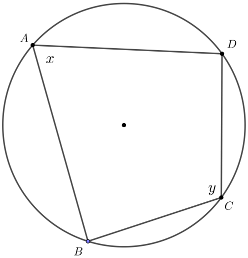
Cyclic Quadrilaterals

A quadrilateral that has all its vertices lying on the same circle is called a *cyclic quadrilateral*. In our diagram, $ABCD$ is a cyclic quadrilateral.



Another Circle Theorem

Circle Theorem 4: The opposite angles of a cyclic quadrilateral are supplementary. In the diagram, $x + y = 180^\circ$

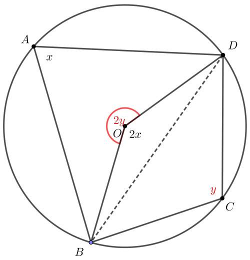


The proof is on the next page.

Another Circle Theorem

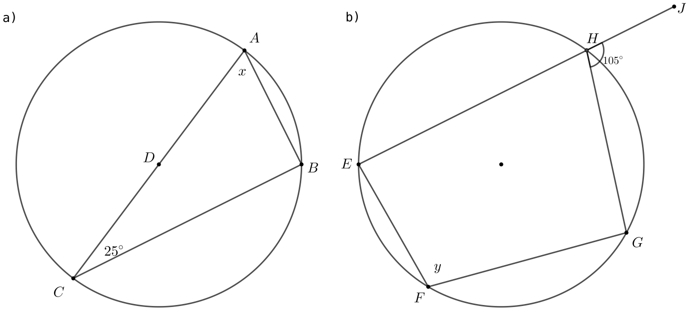
Proof of Circle Theorem 4:

Construct radii BO, DO and chord BD.
$\angle BAD$ is an inscribed angle of chord BD.
The associated central angle is the smaller angle $\angle BOD$.
Therefore, $\angle BOD = 2\angle BAD = 2x$.



Similarly, we can show reflex angle $\angle BOD = 2y$.
Therefore, $2x + 2y = 360°$. and $x + y = 180°$
Therefore, the opposite angles of a cyclic quadrilateral are supplementary.

For each question, find the value of the unknowns. Justify your answers.



Solutions are given on the next page.

Circle Theorems Exercises Solutions

a) Since AB is a diameter, $\angle ABC$ is an inscribed angle and therefore, by Circle Theorem 3 $\angle ABC = 90°$. Now all the angles in a triangle, therefore, $\angle BAC + \angle ABC + \angle ACB = 180$. or $\angle BAC + 90 + 25 = 180$ and it follows $\angle BAC = 65$ Therefore $x = 65°$.

b) Since EHJ is a straight line, then $\angle JHG + \angle EHG = 180$ or $105 + \angle EHG = 180$ and it follows $\angle EHG = 75$. Now, $EFGH$ is a cyclic quadrilateral. From Circle Theorem 4, $\angle EFG + \angle EHG = 180$ or $\angle EFG + 75 = 180$ and it follows that $\angle EFG = 105$. Therefore, $y = 105°$.

Problem Set

You may now work on Problem Set 2.