Math Circles 2021

Complex Numbers, Lesson 2

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Warmup and Review

Here are the key things we discussed last week:

- The numbers 1 and 0 are special, because they are identities (multiplicative and additive, respectively).
- When two numbers combine in an operation to result in the identity for the operation, those numbers are called *inverses*. For example 8 and -8 are additive inverses of each other, while -9 and $-\frac{1}{9}$ are multiplicative inverses of each other.
- Subtraction and division are convenience operations that can just as easily be represented by addition and multiplication using inverses.
- The number represented by the symbol i is defined so that $i^2 = -1$.
- Although i is not actually the same as $\sqrt{-1}$, we can usually safely pretend that it is. We just need to be careful.

Warmup and Review

- All numbers are imaginary, but some are $\mathbb{R}$*eal*, and some are $\mathbb{I}$*maginary*. If $a \in \mathbb{R}$, then $ai \in \mathbb{I}$.
- If we find ourselves in a situation where it would be nice to add a $\mathbb{R}$eal number to an $\mathbb{I}$maginary number, we can do so, and the result is called a $\mathbb{C}$omplex number, which in standard (aka rectangular) form looks like $z = a + bi$.
- For $z = a + bi$, $a, b \in \mathbb{R}$. a is called the real part of z, and b is called the imaginary part of z. We write $a = Re(z)$, and $b = Im(z)$.
- We define a useful property of complex numbers as the *modulus* (or magnitude). The formula is $|z| = \sqrt{a^2 + b^2}$, which is not coincidentally reminiscent of Pythagorean Theorem.

Warmup and Review

Warmup Questions

1. Evaluate $(2+i)(3-5i)$
2. Evaluate $(3-2i)^2$
3. Evaluate $(1+3i)^4$
4. Determine $|z|$ if we know that $z^2-2z+8=0$.

Pause the video and try these questions on your own...

Complex Conjugates and Multiplicative Inverses

Complex conjugates

Given a Complex number $z = a + bi$, we define the *conjugate* of z as $\overline{z} = a - bi$.

Properties of Conjugates

If $z, w \in \mathbb{C}$ then

1. $\overline{z + w} = \overline{z} + \overline{w}$
2. $\overline{zw} = \overline{z}\ \overline{w}$
3. $\overline{\overline{z}} = z$
4. $z + \overline{z} = 2(Re(z)) \in \mathbb{R}$
5. $z - \overline{z} = 2i(Im(z)) \in \mathbb{I}$

Let's prove a few of these properties now...

Complex Conjugates and Multiplicative Inverses (cont'd)

- Complex conjugates have the property that when multiplied together, we always get a $\mathbb{R}$eal number, and a useful one at that:

$$\begin{aligned}(z)(\overline{z}) &= (a+bi)(a-bi) \\ &= a^2-(bi)^2 \\ &= a^2-b^2i^2 \\ &= a^2-b^2(-1) \\ &= a^2+b^2 \\ &= |z|^2 \quad (\in \mathbb{R})\end{aligned}$$

- Note that:
  - If $z \neq 0$ then $|z|^2 \neq 0$.
  - Since $|z|^2 \in \mathbb{R}$, then for $z \neq 0$, $z\left(\frac{1}{|z|^2}\overline{z}\right) = z\overline{z} \div |z|^2 = 1$, which is the multiplicative identity in $\mathbb{C}$.
- $\therefore$ the multiplicative inverse of $z \in \mathbb{C}, z \neq 0$ is $z^{-1} = \frac{1}{|z|^2}\overline{z}$.

Complex Conjugates and Multiplicative Inverses (cont'd)

Example - Demonstration of Multiplicative Inverse

Given $z = 3 + 5i$, we get $\overline{z} = 3 - 5i$, so

$$\begin{aligned}
zz^{-1} &= z\left(\frac{1}{|z|^2}\overline{z}\right) \\
&= (3+5i)\left[\frac{1}{3^2+5^2}(3-5i)\right] \\
&= (3+5i)\left(\frac{3}{34}-\frac{5}{34}i\right) \\
&= \frac{9}{34}-\frac{15}{34}i+\frac{15}{34}i-\frac{25}{34}i^2 \\
&= \frac{9}{34}+\frac{25}{34} \\
&= \frac{34}{34} \\
&= 1
\end{aligned}$$

Complex Conjugates and Multiplicative Inverses (cont'd)

- Just like an additive inverse gives us a way to subtract, a multiplicative inverse gives us a way to "divide".
- Technically there is no division with $\mathbb{C}$omplex numbers, though we often see and accept "lazy" notation.
- So z^{-1} is often written as $\dfrac{\overline{z}}{|z|^2}$

Multiplicative Inverses (cont'd)

Example 2 - Demonstration of "Division"

Given $x = 7 - 3i$ and $y = 5 + 4i$, $x \div y$, or $\dfrac{x}{y}$, is technically not defined, but essentially means the same thing as xy^{-1}.

$$\begin{aligned}
\frac{x}{y} &= xy^{-1} \\
&= \frac{x\overline{y}}{|y|^2} && \leftarrow \text{a more convenient way to express the above} \\
&= \frac{7-3i}{5+4i} \times \frac{5-4i}{5-4i} && \leftarrow \text{a more convenient way to think of the calculation} \\
&= \frac{35 - 28i - 15i + 12i^2}{5^2 + 4^2} \\
&= \frac{1}{41}(23 - 43i)
\end{aligned}$$

For convenience, this result may be written as $\dfrac{23 - 43i}{41}$.

Practice

Try these examples

1. Given $z = -4 - 5i$, determine z^{-1}.
2. If $x = 3 + 7i$ and $y = -2 + 4i$, write $\dfrac{x}{y}$ in standard form.

Pause the video and work on these...

Exponents

To think about exponents, we need some definitions. Let $z \in \mathbb{C}$. We define:

- $z^0 = 1$
- $z^1 = z$
- $z^{k+1} = z^k z$, $k \in \mathbb{N}$
- Note that this definition does not help us for non-natural exponents. We'll get to that.

Exponents

Example

Find a $\mathbb{R}$eal solution to

$$6z^3 + \left(1 + 3\sqrt{2}i\right) z^2 - \left(11 - 2\sqrt{2}i\right) z - 6 = 0$$

This looks much harder than it is!

Powers of i

Consider the following progression:

n	i^n	Result
0	i^0	1
1	i^1	i
2	i^2	-1
3	$i^3 = i^2 \cdot i$	$-i$
4	$i^4 = i^3 \cdot i$	1
5	$i^5 = i^4 \cdot i$	i
6	$i^6 = i^5 \cdot i$	-1
.	.	.
.	.	.
.	.	.
[0] in $\mathbb{Z}_4$	i^{4k}	1
[1] in $\mathbb{Z}_4$	i^{1+4k}	i
[2] in $\mathbb{Z}_4$	i^{2+4k}	-1
[3] in $\mathbb{Z}_4$	i^{3+4k}	$-i$

Cool! The powers of i repeat in cycles of 4, so that for any $n \in \mathbb{N}$, $i^n \in \{1, i, -1, -i\}$.

Negative Integer Exponents

- What about negative integer exponents?
- We want i^{-1} to be the multiplicative inverse of i.
- If the pattern we see held for negative integer exponents, then i^{-1} would be $-i$.
- How fortunate for us then that $i(-i) = -i^2 = 1$!
- Therefore we can happily conclude that $i^{-1} = -i$.
- Good news!
  - It means the cycle we saw just before actually goes backwards as well, and
  - We can convert any negative integer power of i to an expression with a positive exponent, as follows.

Negative Integer Exponents (cont'd)

Let $x > 0$, where $x \in \mathbb{Z}$. Then

$$\begin{aligned} i^{-x} &= \left(i^{-1}\right)^x \\ &= (-i)^x \quad (\text{since } i^{-1} = -i) \\ &= (-1 \times i)^x \\ &= (-1)^x i^x \end{aligned}$$

- We have so far only considered integer values for x, but
- this result actually holds for any $\mathbb{R}$eal value of x, although
- when x is not an integer things do get more ... interesting.
- Very handy!

Example

$$\begin{aligned} i^{-18} &= (-1)^{18} i^{18} \\ &= (1)(-1) \\ &= -1 \end{aligned}$$

Practice

Work on questions 1 - 4 in section 4.11.