Math Circles 2021

Intro to Complex Numbers
Fall, 2021

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Today

Today's Adventure

- Who is Rich Dlin?
- What do I need to know before we start?
  - Be able to use algebra to solve linear equations.
  - Be able to isolate a given variable in a linear equation containing more than one variable.
  - Know the meaning of "reciprocal".

Laying a Foundation For Complex Numbers

So what are these Complex Number thingies anyway?

- Some of you may have heard of them already, or at least heard whispers in the hallways ..
- We'll start with some "trivial exploration" ...

Solve for x

$$\frac{3x+8}{5} = 4$$

(I know ... shockingly difficult)

- Obviously(?) the solution is $x = 4$.
- But once we start using *algorithms* learned in some math class, we introduce significant yet abstract notions that are rarely if ever addressed.
- So let's address them!

Identities and Inverses

Identities

The number 0 is called the *Identity Element* under addition, because for any x, $x + 0 = 0 + x = x$.
Similarly, the number 1 is called the *Identity Element* under multiplication, because for any x, $1 \cdot x = x \cdot 1 = x$.

Inverses

If a is the identity element for an operation $\diamond$, then if $x \diamond y = y \diamond x = a$, we say that x and y are *inverses* of each other under $\diamond$.

Examples:

- The inverse of 5 under multiplication is $\frac{1}{5}$
- The inverse of 5 under addition is -5.
- Subtraction and division are really just addition and multiplication, using the appropriate inverse.
- In this way. "$-$" and "$\div$" are really just convenience operators.
- For example $5 - 2 = 5 + (-2)$, and $7 \div 9 = 7 \times \frac{1}{9}$.
- This is extremely significant.

Solving Equations

Let's solve some equations. But let's really think while we do.

$$
\begin{aligned}
3a - 7 &= 42 && (1)\\
3a - 7 + 7 &= 42 + 7 && (2)\\
3a + 0 &= 49 && (3)\\
3a &= 49 && (4)
\end{aligned}
\qquad
\begin{aligned}
\frac{1}{3} \times 3a &= \frac{1}{3} \times 49 && (5)\\
1 \times a &= \frac{49}{3} && (6)\\
a &= \frac{49}{3} && (7)
\end{aligned}
$$

- The above is really a logical argument
- It begins with the assumption that "There exists $a \in \mathbb{R}$, such that 7 less than triple the value of a results in 42".
- So we have proven this implication

  If $3a - 7 = 42$ **then** $a = \frac{49}{3}$.

  (Interesting thought: What are we proving when we "check the answer"?)
- We did it by using some fundamental numeric concepts, not the least of which is the utility of identity elements and inverses.

Universe of Discourse

Universe of Discourse

In math, the *Universe of Discourse* is the set outside which no possibilities are considered.

Examples

- When deciding how many seats to put in a movie theatre, the universe of discourse is the $\mathbb{N}$atural Numbers: $\{1, 2, 3, 4, 5, \ldots\}$.
- When timing a 100 m sprint, the universe of discourse is $\mathbb{Q}^+$ (the positive rational numbers), which is to say, numbers that can be written as a decimal value that either terminates or repeats in cycles.
- When solving the equation above, we assumed the universe of discourse was the $\mathbb{R}$eal numbers. If it had been the integers, we would have found no solution.

You try

In section 1.5 of the handout, solve questions 4, 9 and 10 using the $\mathbb{R}$eal numbers as the universe of discourse.

Solving Harder Equations

Baby's First Quadratic Equation

$$\begin{aligned} 3x^2 - 7 &= 41 \\ 3x^2 &= 48 \\ x^2 &= 16 \end{aligned}$$

- So we have $x^2 = 16$
- Should we use square root to determine the solution?
- Many people deduce $x = 4$. Why?
- Because they intuitively use a universe of discourse of non-negative integers.
- If we expand the universe of discourse to $\mathbb{R}$, (which requires some imagination) we get $x = \pm 4$.
- Many students then say something like "16 has two square roots".
- But 16 does not have two distinct square roots.
  - (entirely because square root is not *defined* that way)!

Solving Harder Equations

- In fact the solution is the logical statement $(x = \sqrt{16})$ OR $(x = -\sqrt{16})$.
- We use properties of multiplication and negative numbers to deduce the second root of the equation, namely -4.
- This is *not* intuitive. At first.
- Understanding this requires a background in "understanding" negative numbers.
- Interesting! Do we really "understand" negative numbers?
  - Are negative numbers intuitive?
  - We call them $\mathbb{R}$eal, but are they truly realistic?

Thinking More Deeply About Equations

Now let's turn to a somewhat mystifying equation:

Solve For x

$$x^2 = -1$$

- Is there a solution?
- In the Real numbers, there is no number capable of squaring to a negative.
- Therefore, no $\mathbb{R}$eal solution.
- But wait! We have already shown that we are capable of imagining unrealistic numbers ...
- ... so surely we can imagine a universe of discourse with a solution?

Imagining Numbers

Consider this:

- Is it the number three?
- Most people are quick to say yes.
- However, it is not so simple.
- The actual answer is NO.

It is a *symbol* that we have all agreed would represent the number three, so that when we see it, or write it, it is tied to the notion of *three* in our minds.

But what is three?

Can you define this number?

Imagining Numbers

Try to Define "Three"

Pause this video and write a definition for the number three, that doesn't rely on understanding what three is before the definition is read.

My Attempt to Define "Three"

A counting concept used to represent the total number of elements in a set which could be subdivided into non-empty sets, each containing fewer elements than the original, exactly one of which could be subdivided again in the same manner, none of which could be subdivided in the same manner again.

- This is cumbersome
- A clever eye will realize that this definition depended very much on the definitions of "one" and "zero"
- With these numbers already defined, the definition for three could be made more concise.

Imagining Numbers

Definition of 3

Three is a counting concept used to represent a quantity of objects that is one more than one more than one, and not more than that.

In other words ...

$$1+1+1=3$$

... but what is 1?

If you think you can write a good definition of the number 1, post it on the Piazza board!

$\mathbb{I}$maging Numbers

Definition of i

i is a symbol to represent the number that squares to -1. That is, $i^2 = -1$.

Note: Despite numerous humorous t-shirts, mugs, and beach blankets, it is not the case that $\sqrt{-1} = i$. In the Real number system, the square root function is defined only on non-negative numbers.
Additionally, if it were the case that $\sqrt{-1} = i$, then we would have

$$\begin{aligned} \sqrt{-1} &= i \\ (-1)^{\frac{1}{2}} &= i \\ \left[(-1)^{\frac{1}{2}}\right]^2 &= i^2 \\ \left[(-1)^2\right]^{\frac{1}{2}} &= i^2 \\ [1]^{\frac{1}{2}} &= i^2 \\ 1 &= -1 \end{aligned}$$

which would be ... problematic.

$\mathbb{I}$maging Numbers

Some rules for i:

- i has magnitude 1. (I.e., $|i| = 1$).
- i can be scaled using Real numbers, via scalar multiplication. E.g., $3i$, where $|3i| = 3$.
- We define a new number set, the Imaginary numbers, as $\mathbb{I} = \{ai | a \in \mathbb{R}, i^2 = -1\}$.
- This set is unfortunately named, since it implies that to conceive of it requires proprietary implementation of imagination over previously defined number sets.
- Adding two Imaginary numbers works the way we would hope: For all $a, b \in \mathbb{R}, ai + bi = (a + b)i$.
- $\mathbb{I}$ and $\mathbb{R}$ share exactly one number, which is the number zero.
- Negatives work the way we would hope: For all $a \in \mathbb{R}, ai + (-a)i = 0$. i.e., the additive inverse of ai is $(-a)i$.
- To multiply two Imaginary numbers, multiply the scalars and square the i. For example $(3i)(2i) = 6i^2 = -6$.

Back to Equations

Once Imaginary numbers have been defined, and we can solve equations like $x^2 = -9$, we quickly want to use this tool to solve harder quadratic equations that previously seemed to have no solution. Here's an example.

Solve for x

$$\begin{aligned} x^2 - 6x + 10 &= 0 \\ x^2 - 6x &= -10 \\ x^2 - 6x + 9 &= -1 \\ (x-3)^2 &= -1 \end{aligned}$$

- This yields two possibilities: $x - 3 = i$ or $x - 3 = -i$.
- We would love to add 3 to both sides. But we don't have any way to add non-zero Real numbers to non-zero Imaginary numbers.
- So we define a way! And in so doing, define a new universe of discourse, loaded with awesomalitiness.

$\mathbb{C}$omplex Numbers

Definition of $\mathbb{C}$

The set of Complex numbers is defined as $\mathbb{C} = \{a + bi | a, b \in \mathbb{R}, i^2 = -1\}$. The Complex number $z = x + yi$ is said to be in *standard form* if $x, y \in \mathbb{R}$ (this is also known as *rectangular form*).

- $\mathbb{C}$ is a set that contains all of $\mathbb{R}$, and all of $\mathbb{I}$, as well as numbers that are not part of either.
- If $z = a + bi$, a is called the *Real part* of z and b is called the *Imaginary part* of z
  - We write $a = \Re(z)$ or $a = Re(z)$, and $b = \Im(z)$ or $b = Im(z)$.
- Adding, multiplying and negating (therefore subtracting) all work as we would hope, honouring the definitions in $\mathbb{R}$ and $\mathbb{I}$.

E.g.,
$$\begin{aligned}(3+7i)(-2+5i) &= (3)(-2)+(3)(5i)+(-2)(7i)+(7i)(5i)\\ &= -6+15i-14i+35i^2\\ &= -6-35+i\\ &= -41+i\end{aligned}$$

$\mathbb{C}$omplex Numbers

- In general:
  - Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$
  - Multiplication: $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$
- The *modulus* (aka magnitude) of a Complex number $z = a + bi$ is defined as $|z| = \sqrt{a^2 + b^2}$.
  - Note this is perfectly consistent for numbers that are strictly Real or strictly Imaginary.
  - It is no coincidence that this looks like The Greek Triangle Theorem ...
  - or Pythagorean Theorem ... whichever name you like to call it.
- E.g., $|3 + 7i| = \sqrt{3^2 + 7^2} = \sqrt{58}$

Let's Put This All to Work!

Homework Exercises

In the booklet, work on the exercises Section 2.5 (1 - 7) and Section 3.3 (1 - 7).