Grade 11–12 Math Circles
October 27th, 2021
Lesson 1: Rational and Irrational Numbers

§1 Introduction

Rational numbers are one of the most fundamental objects of mathematics. If we let

$$\mathbb{N} = \{1, 2, 3, \ldots\} \quad \text{and} \quad \mathbb{Z} = \{\ldots - 2, -1, 0, 1, 2, \ldots\}$$

denote the natural numbers and the integers, then the set of rational numbers, $\mathbb{Q}$, consists of all quotients of the form $\frac{m}{n}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Thus, for example, -3, 0, $\frac{4}{3}$ and $\frac{1}{7}$ are *rational*, while numbers like $\sqrt{2}$, $\log_2 3$ and $\cos(1^\circ)$ are *irrational*...or maybe not? How do we *know* for certain that a particular number, like π, cannot be written in the form $\frac{m}{n}$ for some non-zero integers m and n? It turns out that this question is rather difficult. For example, only in the 20th century it was proved that $\sqrt{2}^{\sqrt{2}} \approx 1.63$ is irrational, and it is still not known whether $\pi^\pi \approx 36.46$ is irrational (or not). In this lesson, we will learn about proofs of irrationality, practice some of them, and then discuss various open problems. Perhaps, some day you will solve one of them, thus inscribing your name into the history of mathematics!

§2 The First Known Irrational Number

The first ever irrational number, $\sqrt{2}$, was discovered in Ancient Greece, and the legend says that the person who proved its irrationality payed for the discovery with his life. Pythagoreans believed that numbers were the essence of all things and, of course, the fact that $\sqrt{2}$ is irrational shook the foundations of their philosophy. Hence they've taken the pledge to keep the proof outlined below in secret.

Theorem 1

The number $\sqrt{2}$ is irrational.

Proof. Assume, for a contradiction, that $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{m}{n}$ for some integer m and positive integer n. Certainly, we can assume that the fraction $\frac{m}{n}$ is written in lowest terms. But then,

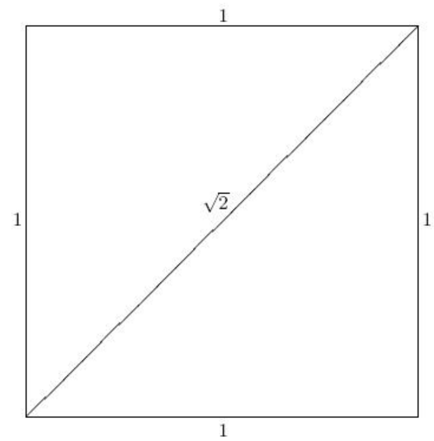

Figure 1: The length of the diagonal of a square with side length one is $\sqrt{2}$

since it is written in lowest terms, m and n cannot both be even. Thus, either m is odd or n is odd. Let us remember this fact, as it will become very important to us later.

Let us multiply both sides of the equation $\sqrt{2} = \frac{m}{n}$ by n so to obtain $\sqrt{2} \cdot n = m$. But then $\left(\sqrt{2} \cdot n\right)^2 = m^2$, and so $2n^2 = m^2$. Thus we see that the number m^2 is even. Since $m \cdot m$ is even, we see that m must be even as well. Hence there must exist an integer k such that $m = 2k$.

Now, let us substitute $m = 2k$ into the equation $2n^2 = m^2$, so to obtain $2n^2 = 4k^2$. Dividing both sides by 2, we obtain $n^2 = 2k^2$. Thus we see that the number n^2 is even. Since $n \cdot n$ is even, we see that n must be even as well. Of course, the fact that m and n are both even contradicts our observation that either m is odd or n is odd. Thus we see that $\sqrt{2}$ is irrational. □

Exercise 1 (Every p-th Root of 2 Is Irrational)

Prove that, for every integer $p \geq 2$, the number $\sqrt[p]{2}$ is irrational.

Exercise 1 Solution

Let $p \geq 2$ be an integer and assume for a contradiction that $\sqrt[p]{2}$ is rational. Then $\sqrt[p]{2} = \frac{m}{n}$ for some integer m and positive integer n. Certainly, we can assume that the fraction $\frac{m}{n}$ is written in lowest terms. But then, since it is written in lowest terms, m and n cannot both be even. Thus, either m is odd or n is odd. Let us remember this fact, as it will become very important to us later.

Let us multiply both sides of the equation $\sqrt[p]{2} = \frac{m}{n}$ by n so to obtain $\sqrt[p]{2} \cdot n = m$. But then $\left(\sqrt[p]{2} \cdot n\right)^p = m^p$, and so $2n^p = m^p$. Thus we see that the number m^p is even. Since m^p is even, we see that m must be even as well. Hence there must exist an integer k such that $m = 2k$. Now, let us substitute $m = 2k$ into the equation $2n^p = m^p$, so to obtain $2n^p = 2^p k^p$. Dividing both sides by 2, we obtain $n^p = 2^{p-1}k^p$. Thus we see that the number n^p is even. Since n^p is even, we see that n must be even as well. Of course, the fact that m and n are even contradicts our observation that either m is odd or n is odd. Thus we see that $\sqrt[p]{2}$ is irrational.

Plato's mathematics tutor, Theodorus of Cerene (born 470 B.C.), is said to extend the above irrationality result to various other numbers, including $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$ and $\sqrt{7}$, while another student of Theodorus, Theaetetus of Athens (415–369 B.C.), proved the following more general result, which we leave to you as a (challenging!) exercise.

Exercise 2 (Theaetetus' Theorem)

Let q be a positive integer. Prove that if q is not a perfect square, then the number $\sqrt{q}$ is irrational.

Hint: Let t^2 be the largest perfect square that divides q, and write $q = s \cdot t^2$ for some positive integer s. Explain why $s > 1$ and why the only perfect square that divides s is 1.

Exercise 2 Solution

Let t^2, with $t \in \mathbb{N}$, denote the largest perfect square that divides q. Then $q = s \cdot t^2$ for some positive integer s. Notice that $s > 1$, for otherwise q would be a perfect square. Further, if a perfect square r^2 divides s, then $(rt)^2$ divides q, and since t^2 is the largest perfect square that divides q, we must have $r^2 = 1$. Thus, the largest perfect square that divides s is 1.

Now, assume for a contradiction that $\sqrt{q} = t\sqrt{s}$ is rational. Then $\sqrt{s} = \frac{m}{n}$ for some integer m and positive integer n. Certainly, we can assume that the fraction $\frac{m}{n}$ is written in lowest terms. Thus, m and n have no divisors in common.

Let us multiply both sides of the equation $\sqrt{s} = \frac{m}{n}$ by n so to obtain $\sqrt{s} \cdot n = m$. But then $\left(\sqrt{s} \cdot n\right)^2 = m^2$, and so $sn^2 = m^2$. Since s divides m^2, we see that s and m share a common prime divisor p. Hence there must exist an integer k such that $m = pk$.

Now, let us substitute $m = pk$ into the equation $sn^2 = m^2$, so to obtain $sn^2 = p^2k^2$. Dividing both sides by p, we obtain $(s/p)n^2 = pk^2$. Since p does not divide s/p, it must be the case that

p divides n^2. Since p is prime, we see that n is divisible by p, which means that m and n share a common divisor. Of course, this contradicts the fact that m and n have no divisors in common.

Some irrationality results follow from certain properties of the rationals. For example, you may know that if r and s are both rational numbers, then

- $r + s$ is rational;
- $r - s$ is rational;
- $r \cdot s$ is rational; and
- r/s is rational, provided that $s \neq 0$.

Try using these properties to solve the following exercise.

Exercise 3 (The Golden Ratio Is Irrational)

Prove each of the following:

(a) Prove that if a is rational and b is irrational, then $a + b$ is irrational.

(b) Prove that if $a \neq 0$ is rational and b is irrational, then ab is irrational.

(c) Use Exercise 2, as well as Parts (a) and (b), to prove that the golden ratio $\frac{1+\sqrt{5}}{2}$ is irrational.

Exercise 3 Solution

(a) Assume, for a contradiction, that $a + b$ is rational. Then $b = (a + b) - a$, and since the difference of two rational numbers is rational, it must be the case that b is rational. This contradicts our assumption that b is irrational.

(b) Assume, for a contradiction, that ab is rational. Since $a \neq 0$, we see that $b = \frac{ab}{a}$, and since the ratio of two rational numbers is rational, it must be the case that b is rational. This contradicts our assumption that b is irrational.

(c) Since 5 is not a perfect square, it follows from Exercise 2 that $\sqrt{5}$ is irrational. Since $\sqrt{5}$ is irrational, it follows from Part (a) that $1 + \sqrt{5}$ is irrational. Since $1 + \sqrt{5}$ is irrational, then it follows from Part (b) that $\frac{1}{2} \cdot (1 + \sqrt{5})$ is irrational.

If you would like to learn about mathematics in Ancient Greece and how Eudoxus of Cnidos (408–355

B.C.) resolved the crisis in the foundations of mathematics, consider reading Chapter 3 in Burton's *History of Mathematics* [1].

§3 Logarithms and Irrationality

Apart from square roots, other objects for which it is possible to establish irrationality results are *logarithms*. Recall that, if a and b are positive real numbers, with $a \neq 1$, then we refer to the number x such that

$$a^x = b$$

as the *logarithm of b in base a*, and denote it by $x = \log_a b$. For example,

- since $2^2 = 4$, we see that $\log_2 4 = 2$;
- since $3^{-2} = \frac{1}{9}$, we see that $\log_3 \frac{1}{9} = -2$;
- since $25^{1/2} = 5$, we see that $\log_{25} 5 = \frac{1}{2}$.

In each of the examples above, the value of logarithms are especially nice, but it's not always that way. For example, while the equation $2^x = 3$ does have a solution (see Figure 2 below), it seems that it can only be *approximated*, but not written down explicitly as a rational number. In fact, up to 7 decimal places, we have $\log_2 3 \approx 1.5849625$.

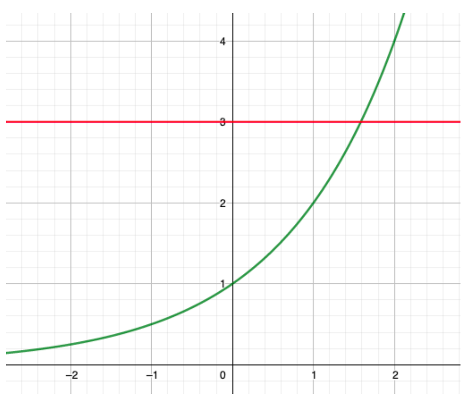

Figure 2: The graphs of $f(x) = 2^x$ and $g(x) = 3$ intersect at $x \approx 1.5849625$

If a, b and c are real numbers, with $a \neq 1$, then the logarithmic functions satisfy the following important properties:

- $\log_a a = 1$
- $a^{\log_a b} = b$

- $\log_a(bc) = \log_a b + \log_a c$

All of these properties follows from the exponent rules, and we will use these properties to solve exercises and prove theorems outlined below.

In what follows, we will restrict our attention to the case when $a \geq 2$ and $b \geq 2$ are integers. Notice how $\log_2 3$ seems to be irrational (we will prove this in Theorem 2), while $\log_2 4 = 2$ is rational. Let us explore what values of a and b guarantee that $\log_a b$ is irrational.

Theorem 2

The number $\log_2 3$ is irrational.

Proof. Suppose not and $\log_2 3 = \frac{m}{n}$ for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Since $\log_2 3 \approx 1.58 > 0$, we see that $m \in \mathbb{N}$. Thus,

$$n \log_2 3 = m$$

$$2^{n \log_2 3} = 2^m$$

$$\left(2^{\log_2 3}\right)^n = 2^m$$

$$3^n = 2^m$$

Since the number on the left-hand side is odd while the number on the right-hand side is even, we reach a contradiction. Thus, $\log_2 3$ is irrational. □

Exercise 4

Prove that the number $\log_3(15)$ is irrational.

Exercise 4 Solution

First, we prove that $\log_3 5$ is irrational. Assume, for a contradiction, that this is not the case, and $\log_3 5 = \frac{m}{n}$ for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Since $\log_3 5 \approx 1.46 > 0$, we see that $m \in \mathbb{N}$. Thus,

$$n \log_3 5 = m$$

$$3^{n \log_3 5} = 3^m$$

$$\left(3^{\log_3 5}\right)^n = 3^m$$

$$5^n = 3^m$$

Since the number on the left-hand side is divisible by 5, while the number on the right-hand side is not divisible by 5, we reach a contradiction.
Now, since $\log_3 5$ is irrational, it follows from Part (a) of Exercise 2 that

$$1 + \log_3 5 = \log_3 3 + \log_3 5 = \log_3(15)$$

is irrational.

The following result provides a useful criterion for determining whether the number $\log_a b$ is irrational.

Exercise 5

Let $a \geq 2$ and $b \geq 2$ be integers. Prove that if $b \neq \sqrt[n]{a^m}$ for all $m, n \in \mathbb{N}$, then $\log_a b$ is irrational.

Exercise 5 Solution

Let $a \geq 2$ and $b \geq 2$ be integers such that $b \neq \sqrt[n]{a^m}$ for all $m, n \in \mathbb{N}$. Assume, for a contradiction, that $\log_a b$ is rational, and so $\log_a b = \frac{p}{q}$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Since $b \geq 2$, we see that $\log_a b > 0$, and so $m \in \mathbb{N}$. Thus,

$$q \log_a b = p$$

$$a^{q \log_a b} = a^p$$

$$\left(a^{\log_a b}\right)^q = a^p$$

$$b^q = a^p$$

$$b = \sqrt[q]{a^p}$$

This observation contradicts our assumption that $b \neq \sqrt[n]{a^m}$ for all $m, n \in \mathbb{N}$.

If you would like to learn more about logarithmic functions and their properties, consider completing the module on Advanced Functions and Pre-Calculus in the CEMC Courseware.

§4 Irrationality of Certain Trigonometric Quantities

We've explored irrationality of radicals and logarithms. What about trigonometric quantities? In this section, we will prove the following theorem.

Theorem 3

Let α be a real number. If $\cos\alpha$ is rational, then $\cos(n\alpha)$ is rational for every positive integer n.

This result gives us a useful criterion for determining some irrational values of $\cos\alpha$. More precisely, if we can show that, for a particular angle α, the value $\cos(n\alpha)$ is irrational for some positive integer n, then $\cos\alpha$ must also be irrational. Try applying this criterion in the following exercise.

Exercise 6

Use Theorem 3 to prove that $\cos(1^\circ)$ is irrational.

Exercise 6 Solution

Let $\alpha = 1^\circ$, and assume that $\cos\alpha$ is rational. Take $n = 30$. Then it follows from Theorem 3 that $\cos(n\alpha) = \cos(30^\circ) = \frac{\sqrt{3}}{2}$ is rational, which is false. Thus, $\cos(1^\circ)$ is irrational.

We will now turn our attention to the proof of Theorem 3. For this purpose, we defined so-called *Chebyshev polynomials* as follows:

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_n(x) = (2x)T_{n-1}(x) - T_{n-2}(x)$$

for every integer $n \geq 2$. Thus, for example, we see that

$$\begin{aligned} T_2(x) &= (2x)T_1(x) - T_0(x) \\ &= (2x) \cdot x - 1 \\ &= 2x^2 - 1 \end{aligned}$$

and

$$\begin{aligned} T_3(x) &= (2x)T_2(x) - T_1(x) \\ &= (2x)(2x^2 - 1) - x \\ &= 4x^3 - 3x \end{aligned}$$

Exercise 7

Determine $T_4(x)$ and $T_5(x)$.

Exercise 7 Solution

We have

$$\begin{aligned} T_4(x) &= (2x)T_3(x) - T_2(x) \\ &= (2x)(4x^3 - 3x) - (2x^2 - 1) \\ &= 8x^4 - 8x^2 + 1 \end{aligned}$$

and

$$\begin{aligned} T_5(x) &= (2x)T_4(x) - T_3(x) \\ &= (2x)(8x^4 - 8x^2 + 1) - (4x^3 - 3x) \\ &= 16x^5 - 20x^3 + 5x \end{aligned}$$

As the following theorem shows, Chebyshev polynomials are intimately connected to trigonometric quantities.

Theorem 4

For every real number α and for every non-negative integer n, $T_n(\cos\alpha) = \cos(n\alpha)$.

Proof. Let α be a real number. Observe that, for $n = 0$,

$$T_0(\cos\alpha) = 1 = \cos(0 \cdot \alpha)$$

so the statement is true in this case. Further, for $n = 1$,

$$T_1(\cos\alpha) = \cos\alpha = \cos(1\cdot\alpha)$$

so once again the statement is true.

Next, let $n \geq 2$ be an integer, and recall the trigonometric identities

$$\begin{aligned}\cos(a-b) &= \cos a\ \cos b\ +\ \sin a\ \sin b\\ \cos(a+b) &= \cos a\ \cos b\ -\ \sin a\ \sin b\end{aligned}$$

which hold for all real numbers a and b. Adding these two equalities together, we obtain the identity

$$\cos(a+b) + \cos(a-b) = 2\cos a\ \cos b$$

Taking $a = (n-1)\alpha$ and $b = \alpha$, we find that

$$\cos(n\alpha) + \cos((n-2)\alpha) = (2\cos\alpha)\cdot\cos((n-1)\alpha)$$

which is equivalent to

$$(2\cos\alpha)\cdot\cos((n-1)\alpha) - \cos((n-2)\alpha) = \cos(n\alpha)$$

Thus,

$$\begin{aligned}T_2(\cos\alpha) &= (2\cos\alpha)T_1(\cos\alpha) - T_0(\cos\alpha)\\ &= (2\cos\alpha)\cdot\cos(\alpha) - \cos(0)\\ &= \cos(2\alpha)\end{aligned}$$

$$\begin{aligned}T_3(\cos\alpha) &= (2\cos\alpha)T_2(\cos\alpha) - T_1(\cos\alpha)\\ &= (2\cos\alpha)\cdot\cos(2\alpha) - \cos(\alpha)\\ &= \cos(3\alpha)\end{aligned}$$

Proceeding in the same fashion for $T_4(x), T_5(x), \ldots$, the result follows. $\square$

We are now ready to prove the main theorem of this section, Theorem 3.

Proof of Theorem 3. Let α be a real number, and assume that $\cos\alpha$ is rational. Then it follows from Theorem 4 that, for every positive integer n, $\cos(n\alpha) = T_n(\cos\alpha)$. In other words, the value $\cos(n\alpha)$ is the result of evaluation of a polynomial $T_n(x)$, all of whose coefficients are rational, at the rational number $\cos\alpha$. Since the product, sum or difference of rational numbers is rational, we see that $T_n(\cos\alpha)$ must be rational. □

Exercise 8

Use Theorem 3 to prove that $\sin(1°)$ is irrational.

Hint: Use the trigonometric identity $\sin(90° + a) = \cos a$, which holds for all real numbers a.

Exercise 8 Solution

Let $\alpha = 1° = \frac{\pi}{180}$, and assume that $\sin\alpha$ is rational. Then it follows from the trigonometric identity $\sin(90° + a) = \cos a$ that

$$\sin\alpha = \sin\left(90° + \alpha - 90°\right) = \cos\left(\alpha - 90°\right) = \cos\left(1° - 90°\right) = \cos\left(-89°\right) = \cos\left(89°\right)$$

Thus, for $\beta = 89°$, the number $\cos\beta$ is rational. Now, let $n = 30$. Then

$$\cos\left(n\beta\right) = \cos\left(2670°\right) = \cos\left(7 \cdot 360° + 150°\right) = \cos\left(150°\right) = -\frac{\sqrt{3}}{2}$$

Note how $\cos(n\beta)$ is irrational. This observation contradicts Theorem 3, which asserts that $\cos(n\beta)$ must be rational.

In Exercise 6 we prove the irrationality of $\cos\alpha$ for the angle $\alpha = 1°$. Notice that, when expressed in radians, this angle is a rational multiple of π, because $1° = \frac{1}{180} \cdot \pi$. In 1946, a Swiss mathematician Hugo Hadwiger explained what angles α that are rational multiples of π have either $\cos\alpha$, or $\sin\alpha$, or $\tan\alpha$ rational.

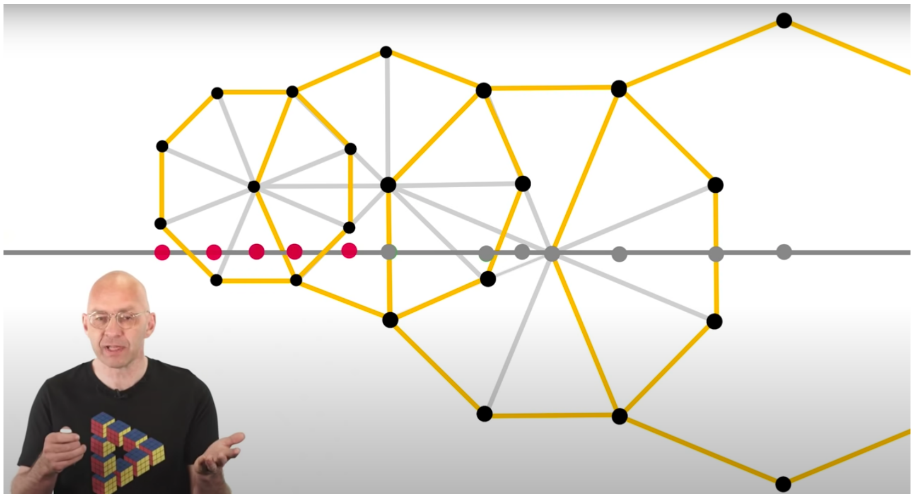

Figure 3: Scene from Mathooger's video on the proof of Hadwiger's Theorem

Theorem 5 (Hadwiger's Theorem)
Let r be a rational number such that $0 \leq r < 2$. Suppose that the angle $\alpha = r\pi$ has the property that either $\cos\alpha$, or $\sin\alpha$, or $\tan\alpha$ is rational. Then,

$$\alpha \in \left\{0,\ \frac{\pi}{6},\ \frac{\pi}{4},\ \frac{\pi}{3},\ \frac{\pi}{2},\ \frac{2\pi}{3},\ \frac{3\pi}{4},\ \frac{5\pi}{6},\ \pi,\ \frac{7\pi}{6},\ \frac{5\pi}{4},\ \frac{4\pi}{3},\ \frac{3\pi}{2},\ \frac{5\pi}{3},\ \frac{7\pi}{4},\ \frac{11\pi}{6}\right\}$$

Proof. A gorgeous proof of this theorem is explained in Burkard Polster's video *What does this prove? Some of the most gorgeous visual "shrink" proofs ever invented* [3]. Make sure to watch this video, as well as other amazing videos from Polster's YouTube channel called *Mathologer*! □

§5 An Interesting Criterion for Irrationality

It turns out that all rational numbers satisfy a useful property, which we will outline below.

Theorem 6

Let r be a rational number. There exists a positive number C, which depends only on r, such that for every rational number $\frac{p}{q} \neq r$, with $q \in \mathbb{N}$, the inequality

$$\left| r - \frac{p}{q} \right| \geq \frac{C}{q}$$

is satisfied.

Proof. Let $r = \frac{m}{n}$ be a rational number, with $n \in \mathbb{N}$, and let $\frac{p}{q} \neq r$ be a rational number. Since $\frac{m}{n} \neq \frac{p}{q}$, we see that $mq - np \neq 0$. Since $mq - np$ is an integer, we conclude that $|mq - np| \geq 1$. Thus,

$$\left| \frac{m}{n} - \frac{p}{q} \right| = \frac{|mq - np|}{nq} \geq \frac{1}{nq} = \frac{C}{q}$$

where $C = \frac{1}{n}$. Notice how the value of C depends only on r. □

Since every rational number must satisfy this criterion, it must be the case that every real number that does **not** satisfy it must immediately be irrational. We summarize this observation as follows.

Criterion for Irrationality

Let r be a real number. If for every positive number C there exists a rational number $\frac{p}{q}$ such that

$$0 < \left| r - \frac{p}{q} \right| < \frac{C}{q}$$

then r is irrational.

This result turns out to be quite useful to prove irrationalities of certain numbers. In 1973, a French mathematician Roger Apéry used this criterion to prove that the number

$$A = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots \approx 1.202$$

is irrational [2]. More precisely, he proved that there exists a positive number D and an infinite sequence of rational numbers $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$, with $1 \le q_1 < q_2 < \ldots$, such that

$$0 < \left| A - \frac{p_n}{q_n} \right| < \frac{D}{q_n^{1.08}}$$

In the following exercise, you are asked to deduce the irrationality of A from Apéry's result.

Exercise 9

Suppose that, for a real number A, there exists a positive number D and an infinite sequence of distinct rational numbers $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$, with $1 \le q_1 < q_2 < \ldots$, such that

$$0 < \left| A - \frac{p_n}{q_n} \right| < \frac{D}{q_n^{1.08}}$$

Use the above criterion for irrationality to prove that A is irrational.

Exercise 9 Solution

Let C be a positive real number. Our goal is to show that there exists a rational number $\frac{p}{q}$ such that

$$0 < \left| \alpha - \frac{p}{q} \right| < \frac{C}{q}$$

Since the sequence $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ found by Apéry is infinite, we can choose $\frac{p}{q} = \frac{p_n}{q_n}$ with the denominator q so large that $q \ge (D/C)^{12.5}$. This inequality is equivalent to $q^{1.08} \ge (D/C)q$. Consequently,

$$0 < \left| A - \frac{p}{q} \right| < \frac{D}{q^{1.08}} \le \frac{D}{(D/C)q} = \frac{C}{q}$$

Today, the number A is known as *Apéry's constant*, and it is still not known whether the number $1 + 2^{-s} + 3^{-s} + 4^{-s} + \cdots$ is irrational for $s = 5$, or $s = 7$, or any odd integer $s \ge 5$. It is known, however, that the number $1 + 2^{-s} + 3^{-s} + 4^{-s} + \cdots$ is irrational for *every* even integer $s \ge 2$. We will talk a little bit about this result in Lesson 2.

While we do know that certain famous numbers, like π (Lambert, 1761) or e^{π} are irrational (Gelfond-Schneider Theorem, 1934), for many numbers, like $e\pi$, $\pi + e$, π^{π}, 2^e, π^e, $\pi^{\sqrt{2}}$ or $\ln \pi$, the question still remains open.

References

[1] David M. Burton, *The History of Mathematics, An Introduction*, 7th Edition, McGraw-Hill, 2011.

[2] Alfred van der Poorten, *A proof that Euler missed... Apéry's proof of the irrationality of* $\zeta(3)$, The Mathematical Intelligencer 1, pp. 195–203, 1979.

[3] Burkard Polster, *What does this prove? Some of the most gorgeous visual "shrink" proofs ever invented*, Mathologer video, https://www.youtube.com/watch?v=sDfzCIWpS7Q, 2020.