Grade 11/12 Math Circles
Cardinality II

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Zack Cramer

In our first lesson—Cardinality I—we defined what it meant for two sets to have the same size or *cardinality*. Specifically, sets A and B are said to have the same **cardinality** or be in **one-to-one correspondence** if there exists a bijective function $f : A \to B$. In this case, we write $|A| = |B|$.

We showed that the sets of natural numbers, integers, and rational numbers all have the same cardinality. These are examples of *countable* sets – their elements can be written in a list.

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|.$$

You also showed in Problem Set I that the interval $(0, 1)$ and the entire real number line have the same cardinality, but—as we saw in the notes—this cardinality is strictly larger than that of $\mathbb{N}$. We say that $(0, 1)$ and $\mathbb{R}$ are *uncountable* sets.

$$|(0, 1)| = |\mathbb{R}|.$$

Exercise 1: We introduced terminology to describe sets of different cardinalities. To refresh your memory of these terms, complete the following definitions from our first lesson.

(i) A set A is **finite** if ______________________________.

(ii) A set A is **countably infinite** if ______________________________.

(iii) A set A is **countable** if ______________________________.

(iv) A set A is **uncountable** if ______________________________.

In our live session, we investigated the cardinality of a *union* of sets. Recall that if A and B are sets, then the **union** of A and B is set $A \cup B$ consisting of all elements in A or B (or both.) That is,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

We also showed the following, which may be useful to you on this week's problem set:

Proposition 1:

(i) If A and B are countable sets, then is $A \cup B$ countable.

(ii) More generally, if $A_1, A_2, A_3, \ldots$ is a *countable* collection of *countable* sets, then

$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup A_3 \cup \cdots$$

is countable.

In this lesson will will continue to explore the cardinality of sets. We will learn how to construct new and interesting sets from familiar examples and determine the cardinalities of these sets. Specifically, we'll study the cardinalities of Cartesian products and power sets.

Subsets

We'll begin this section with a definition that you may remember from the Lesson 1 Problem Set.

Definition 1: Let A and B be sets. We say that A is a **subset** of B if every element of A is also an element of B. In this case, we write $A \subseteq B$.

Example 1

1. $\{1, 2\} \subseteq \mathbb{N}$, since $1 \in \mathbb{N}$ and $2 \in \mathbb{N}$. However, $\{0, 1\}$ is not a subset of $\mathbb{N}$ since $0 \notin \mathbb{N}$. In this case, we write $\{0, 1\} \not\subseteq \mathbb{N}$.

2. $\mathbb{N} \subseteq \mathbb{Z}$, since every natural number is also an integer.

3. More generally, $\mathbb{Z}_n \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

4. For every set A, we have $A \subseteq A$.

5. For every set A, the empty set, $\emptyset$, is a subset of A.

If this seems odd, think of it this way: if there were a set A that *didn't* have $\emptyset$ as a subset, it would mean that there is an element of $\emptyset$ that isn't in A. But since $\emptyset$ has no elements, this is impossible. So indeed, $\emptyset \subseteq A$.

Note: The notation "$\subseteq$" used for subsets should remind you of the notation "$\leq$" used for comparing the size of two real numbers. This is intentional – $A \subseteq B$ means that B contains everything in A and maybe more, hence A is, in some sense, less than or equal to B.

Exercise 2: Is it possible for a set to have the same cardinality as one of its subsets? The answer is "yes", since for any set A, we have $A \subseteq A$ and, of course, $|A| = |A|$. But what if we insist that the set and subset not be equal? That is...

Do there exist sets A and B such that $A \subseteq B$, $A \neq B$, and $|A| = |B|$?

Does the answer change if we require A and B to be finite sets?

Power Sets

Given a set A, we can define a new set, $\mathcal{P}(A)$, whose elements are all subsets of A. This set is known as the *power set* of A.

Definition 2: The **power set** of a set A is the set consisting of all subsets of A:

$$\mathcal{P}(A) = \{S : S \subseteq A\}.$$

Example 2

(a) Consider the set $A = \{1\}$ containing just one element. This set has only two subsets: $\emptyset$ and A itself. Thus, $\mathcal{P}(A)$ is the 2-element set

$$\mathcal{P}(A) = \{\emptyset, \{1\}\}.$$

(b) If $A = \{0,1\}$, then the subsets of A are $\emptyset$, $\{0\}$, $\{1\}$, and $\{0,1\}$. Thus, $\mathcal{P}(A)$ is the 4-element set

$$\mathcal{P}(A) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}.$$

(c) If $A = \emptyset$, then the only subset of A is A itself, hence $\mathcal{P}(A) = \{\emptyset\}$. Note that the set $\{\emptyset\}$ isn't empty – it's a set containing exactly one element: the empty set. Thus, $|\mathcal{P}(A)| = 1$.

Given the cardinality of a set A, what can be said about the cardinality of $\mathcal{P}(A)$? The above examples provide some insight:

Cardinality of A	0	1	2
Cardinality of $\mathcal{P}(A)$	1	2	4

If you can't quite see the pattern here, try finding the cardinality of $\mathcal{P}(A)$ when $|A| = 3$ or $|A| = 4$.

The following proposition describes the relationship between $|A|$ and $|\mathcal{P}(A)|$ when A is a finite set.

Proposition 2: Let n be a positive integer. If A is a set with $|A| = n$, then $|\mathcal{P}(A)| = 2^n$.

Proof: How many ways are there to construct a subset of A? For each element of A, we have two choices: we can either include it as an element of the subset or not. Since there are 2 choices for each element of A and n elements in total, there are $2 \cdot 2 \cdots 2 = 2^n$ different ways to form a subset of A. Thus, $|\mathcal{P}(A)| = 2^n$. □

For example, Proposition 2 tells us that the power set of $A = \{0,1,2\}$ will have $2^3 = 8$ elements. We can confirm this by listing the elements as follows:

$$\mathcal{P}(A) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$$

Remember, $\mathcal{P}(A)$ is just another set, so we are able to talk about its power set as well. This new power set, $\mathcal{P}(\mathcal{P}(A))$, consists of all subsets of $\mathcal{P}(A)$. Let's check out some examples.

Example 3:

(a) We saw that the power set of $A = \{1\}$ is the 2-element set $\mathcal{P}(A) = \{\emptyset, \{1\}\}$. Thus, the power set $\mathcal{P}(\mathcal{P}(A))$ will have $2^2 = 4$ elements (by Proposition 2) and is given as follows:

$$\mathcal{P}(\mathcal{P}(A)) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\}.$$

(b) The power set of $A = \{0, 1\}$ has $2^2 = 4$ elements, and therefore $|\mathcal{P}(\mathcal{P}(A))| = 2^4 = 16$. As an exercise, try listing the elements of $\mathcal{P}(\mathcal{P}(A))$!

(c) The power set of $A = \{0, 1, 2\}$ has cardinality $2^3 = 8$, and therefore $\mathcal{P}(\mathcal{P}(A))$ contains a whopping $2^8 = 256$ elements!

The above examples show that for finite sets, the cardinality of $\mathcal{P}(A)$ is often quite a bit larger than the cardinality of A itself. It turns out that the same is true for even infinite sets.

To make this idea precise, let A be *any* set (finite or infinite). The power set of A will be at least as large as A since it contains all sets of the form $\{a\}$ where $a \in A$. But can it ever be the case that $|A| = |\mathcal{P}(A)|$? No – the power set is always bigger! This result, due to Cantor, is presented below. Its proof is one of my favourites in all of mathematics!

Cantor's Theorem: For every set A, $|A| \neq |\mathcal{P}(A)|$.

Proof: Consider what it would mean if $|A|$ were, in fact, equal to $|\mathcal{P}(A)|$. In this case, there would exist a bijection $f : A \to \mathcal{P}(A)$ that pairs each element of A with some subset of A, $f(A)$.

To see why the existence of this function f is problematic, consider the set

$$S = \{a \in A : a \notin f(a)\}.$$

This set S is a subset of A – it contains exactly the elements of A that are *not* in the in the set they are matched with by f. Since $S \in \mathcal{P}(A)$ and $f : A \to \mathcal{P}(A)$ is surjective, there is an element $a \in A$ such that $f(a) = S$.

We now consider the following question: is $a \in S$? If so, then by the way we defined S, we know that $a \notin f(a) = S$. This is clearly a contradiction. But if instead $a \notin S$, then again, by the way we defined S, we know that $a \in f(a) = S$ – another contradiction!

Whether $a \in S$ or $a \notin S$, we always reach a contradiction. Since our logic at every stage of the argument was sound, this must mean that our initial assumption—that there exists a bijection $f : A \to \mathcal{P}(A)$—is incorrect. Since no such bijection exists, we conclude that $|A| \neq |\mathcal{P}(A)|$. $\square$

Cantor's Theorem tells us that from any given set A, we can obtain a set with greater cardinality by considering its power set, $\mathcal{P}(A)$. When $A = \mathbb{N}$, for example, Cantor's Theorem says that $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$. But why stop there? We could next consider $\mathcal{P}(\mathcal{P}(\mathbb{N}))$, the set of all subsets of $\mathcal{P}(\mathbb{N})$. According to Cantor, this set has an even larger cardinality than $\mathcal{P}(\mathbb{N})$!

If we continue in this way, we obtain an infinite chain of (infinite) cardinalities!

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < \cdots$$

You may wonder where $|\mathbb{R}|$, our favourite example of an uncountable set, can be found within this chain. It turns out that

$$|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|.$$

To show that this is the case, we'll make use of the fact that every real number can be represented as a decimal in binary using 0's and 1's. For instance, in binary, 0.11 represents $2^{-1} + 2^{-2} = 3/4$, while 0.101 represents $2^{-1} + 2^{-3} = 5/8$. Infinite decimals are also possible, such as $0.101010 = 0.\overline{10}$ to represent $2^{-1} + 2^{-3} + 2^{-5} + \cdots = 2/3$.

Example 4: $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.

Proof: Consider the function f that sends each subset $S \subseteq \mathbb{N}$ to the binary decimal number

$$0.a_1a_2a_3a_4\ldots \quad \text{where} \quad a_n = \begin{cases} 1 & \text{if } n \in S, \\ 0 & \text{if } n \notin S. \end{cases}$$

For example,

$$\begin{aligned} f(\{1,2,4\}) &= 0.11010000\ldots = 0.1101, \\ f(\{2,4,6,8,\ldots\}) &= 0.010101\ldots = 0.\overline{01}, \\ f(\emptyset) &= 0.0000\ldots = 0. \end{aligned}$$

Since each binary decimal represents a real number, f produces real numbers in $[0,1]$ from elements of $\mathcal{P}(\mathbb{N})$. If $f : \mathcal{P}(\mathbb{N}) \to [0,1]$ happens to be bijective, then $|\mathcal{P}(\mathbb{N})| = |[0,1]|$, which we know from Lesson 1 is the same as $|\mathbb{R}|$.

It turns out that f is surjective: it can produce any real number in $[0,1]$ as an output. It is *almost* injective, but not quite. The issue here is that real numbers can have more than one binary expansion. For instance, both 0.1 and $0.011111\ldots = 0.0\overline{1}$ represent the number $1/2$. This means that $f(\{1\}) = \frac{1}{2}$ and $f(\{2,3,4,\ldots\}) = \frac{1}{2}$, so there are cases where different inputs can produce the same output. Uh oh...

Rest assured, it's possible to fix the issue by choosing just one of the binary expansions—either the terminating decimal or non-terminating decimal—to represent each real number in $[0,1]$. We'll accept this as possible and not dwell on the fine details here. Thus, $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$. □

Cartesian Products

Given sets A and B, we can form a new set consisting of all ordered pairs (a,b), where $a \in A$ and $b \in B$. This set is known as the *Cartesian product* of A and B and is denoted $A \times B$.

Definition 3: The **Cartesian product** of sets A and B is the set

$$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}.$$

The notation A^2 is commonly used to denote the Cartesian product $A \times A$.

For instance, if $A = \{1,2\}$ and $B = \{1,2,3\}$, then

$$\begin{aligned} A \times B &= \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}, \\ A^2 = A \times A &= \{(1,1),(1,2),(2,1),(2,2)\} \end{aligned}$$

Notice that the order of the elements in each pair is important: the pairs $(1,2)$ and $(2,1)$, for example, are considered to be distinct.

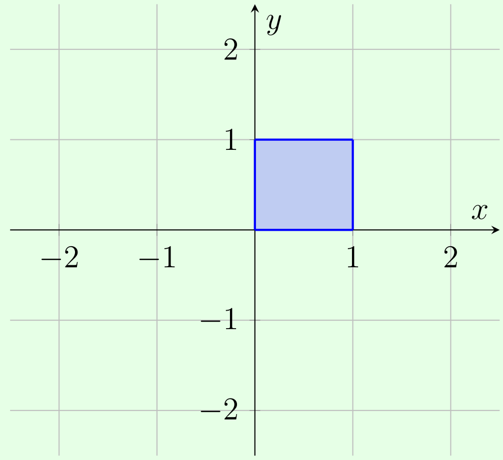
Here are some additional examples involving some familiar sets.

Example 5:

(i) The set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ consists of all pairs (x,y) where $x,y \in \mathbb{R}$. That is, $\mathbb{R}^2$ is the set of all points in the xy-plane.

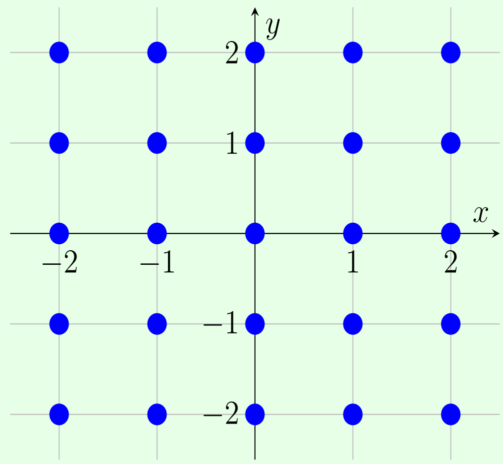
(ii) The set $[0,1]^2 = [0,1] \times [0,1]$ is a subset of $\mathbb{R}^2$. Its elements are the pairs (x, y) such that $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

We can visualize $[0,1]^2$ as the set of all points in the **unit square** shown on the right.



(iii) The set $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ is also a subset of $\mathbb{R}^2$. It consists of all pairs (a, b) where $a, b \in \mathbb{Z}$, such as $(1, 0)$ and $(-2, 1)$.

These points—known as **integer points** or **lattice points**—are shown in blue.

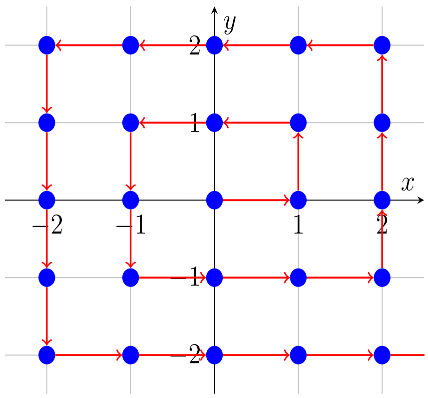


We are interested in understanding the cardinality of a Cartesian product $A \times B$. It's not too hard to see that if A and B are finite sets, so too is $A \times B$. Try to figure out what the cardinality of $A \times B$ might be in this case.

Exercise 3: If A and B are finite sets with $|A| = m$ and $|B| = n$, what is $|A \times B|$?

If instead, A and B are countable sets, what can be said about the cardinality of $A \times B$? The answer may not be so obvious when at least one of A or B is countably infinite. To get a sense of what's going on here, let's check out an example.

Consider the set $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$, which is a Cartesian product of countably infinite sets. We saw above that $\mathbb{Z}^2$ can be visualized as the set of all integer points in the xy-plane. As it happens, this set is countable – the integer points can be listed by following the spiral path from $(0, 0)$ shown below.



$\mathbb{N}$	1	2	3	4	5	$\cdots$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\cdots$
$\mathbb{Z}^2$	$(0,0)$	$(1,0)$	$(1,1)$	$(0,1)$	$(-1,1)$	$\cdots$

Since any countably infinite set is in one-to-one correspondence with $\mathbb{Z}$, perhaps it seems reasonable that a Carestian product of two countable sets will be countable, just as $\mathbb{Z} \times \mathbb{Z}$ is countable. Sure enough, this turns out to be the case.

Proposition 3: If A and B are countable sets, then $A \times B$ is also countable.

Proof: Note that if $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$, as there is no way to form any pair (a, b). Having considered this case, we will assume from here on that $A \neq \emptyset$ and $B \neq \emptyset$.

Since A and B are countable, their elements can be written as lists:

$$\begin{array}{ll} A: & a_1, \quad a_2, \quad a_3, \quad a_4, \quad \ldots \\ B: & b_1, \quad b_2, \quad b_3, \quad b_4, \quad \ldots \end{array}$$

Note that these lists may be finite or infinite.

The elements of $A \times B$ are pairs (a, b) where $a \in A$ and $b \in B$. Therefore, we can consider the pairs whose first entry is a_1, the pairs whose first entry is a_2, and so on; and arrange these pairs in a (possibly infinite) table:

$$\begin{array}{r|cccc} \text{First entry } a_1 & (a_1, b_1) & (a_1, b_2) & (a_1, b_3) & \cdots \\ \text{First entry } a_2 & (a_2, b_1) & (a_2, b_2) & (a_2, b_3) & \cdots \\ \text{First entry } a_3 & (a_3, b_1) & (a_3, b_2) & (a_3, b_3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

From here, how can we show that $A \times B$ is countable? That is, how can we express the elements of $A \times B$ in a (possibly infinite) list? Try to wrap up the argument on your own! $\square$

As an application of Proposition 3, all of the following sets are countable:

$$\mathbb{N}^2, \quad \mathbb{Z}^2, \quad \mathbb{Q}^2, \quad \mathbb{N} \times \mathbb{Z}, \quad \mathbb{N} \times \mathbb{Q}, \quad \mathbb{Z} \times \mathbb{Q}.$$

The case of a Cartesian product involving *uncountable* sets is a little more complicated. Such a product $A \times B$ will certainly be uncountable (provided that A and B are non-empty), but establishing the precise relationship between $|A|$, $|B|$, and $|A \times B|$ is a bit beyond the scope of this lesson.

Rather than addressing Cartesian products of uncountable sets generally, we will focus on the specific example of $\mathbb{R}^2$. It turns out—perhaps surprisingly—that $|\mathbb{R}^2| = |\mathbb{R}|$, meaning that there are the same number of points on the real line as there are in the entire xy-plane!

Example 6: $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$.

Proof: We will begin by showing that $|(0,1) \times (0,1)| = |(0,1)|$. To do this, recall that every element of $(0,1)$ can be written as

$$0.\, a_1 a_2 a_3 \ldots \quad \text{where} \quad a_n \in \{0, 1, 2, \ldots, 9\}.$$

Thus, consider the function $f : (0,1) \times (0,1) \to (0,1)$ that interlaces two decimal numbers to produce a single decimal number:

$$f(0.a_1 a_2 a_3 \ldots, 0.b_1 b_2 b_3 \ldots) = 0.a_1 b_1 a_2 b_2 a_3 b_3 \ldots$$

We have to be a little careful when handling numbers with more than one decimal expansion, just as in Example 4. Once we've agreed on which decimal expansion to use, it is possible to show that our function f is both surjective and injective. Thus, since f is a bijection from $(0,1) \times (0,1)$ to $(0,1)$, these sets have equal cardinality.

To complete the proof that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$, recall from Lesson 1 that $|\mathbb{R}| = |(0,1)|$. It should therefore seem reasonable that $|\mathbb{R} \times \mathbb{R}| = |(0,1) \times (0,1)|$. Consequently,

$$|\mathbb{R} \times \mathbb{R}| = |(0,1) \times (0,1)| = |(0,1)| = |\mathbb{R}|.$$

$\square$

Cartesian Products of Three or More Sets

We can also consider Cartesian products of three or more sets $A_1, A_2, \ldots, A_n$ by extending our definition of $A \times B$ in a very natural way.

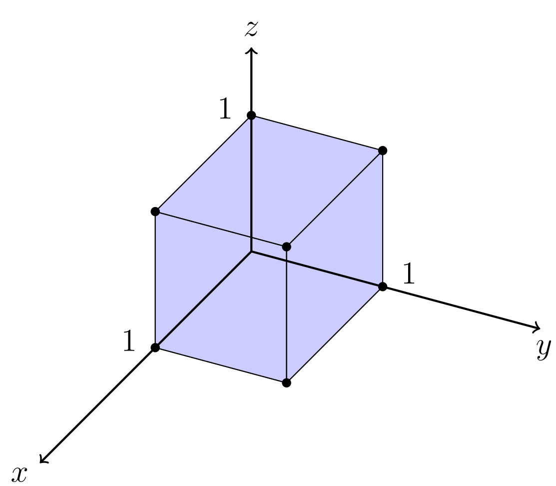
Definition 4: The **Cartesian product** of sets $A_1, A_2, \ldots, A_n$ is the set

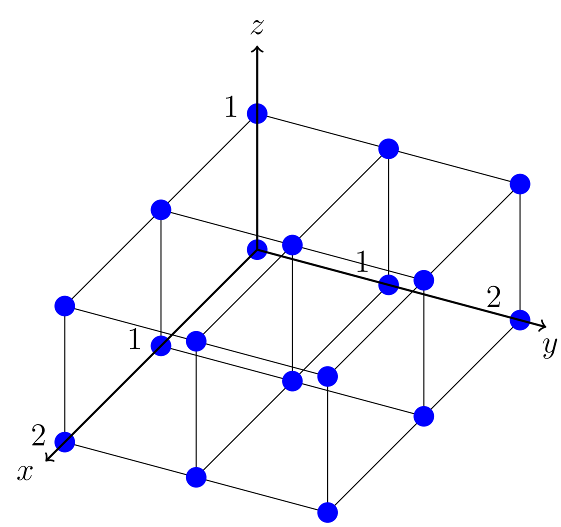
$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

The notation A^n is commonly used to denote the set $\underbrace{A \times A \times \cdots \times A}_{n \text{ times}}$.

As an example, we can visualize $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ as the set of all points in 3D space. In $\mathbb{R}^3$, the location of a point is described relative to three axes: the x-axis, y-axis, and z-axis.

Within $\mathbb{R}^3$, $[0,1]^3 = [0,1] \times [0,1] \times [0,1]$ is the set of points in the **unit cube** shown below (left), while $\mathbb{Z}^3 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ is the set of all **integer points** (or **lattice points**) shown in blue (right).





In the previous section, we argued that if A and B are countable, then so is $A \times B$. Similar arguments show that this result extends to Cartesian products of three or more sets. In particular, the set $\mathbb{Z}^3$ of all integer points in $\mathbb{R}^3$ is countable!

Proposition 4: If $A_1, A_2, \ldots, A_n$ is a finite collection of countable sets, then the Cartesian product

$$A_1 \times A_2 \times \cdots \times A_n$$

is also countable.

Likewise, our proof that $|\mathbb{R}^2| = |\mathbb{R}|$ can also be extended to show that $|\mathbb{R}^n| = |\mathbb{R}|$ for all positive integers n. This means that all of the following sets have the same cardinality:

$$\begin{aligned}
\mathbb{R}^3 &= \text{All of 3D space} \\
\mathbb{R}^2 &= \text{All of 2D space (i.e., all points in the } xy\text{-plane)} \\
\mathbb{R} &= \text{All of 1D space (i.e., all points on the real number line)} \\
[0,1]^3 &= \text{All points in the unit cube} \\
[0,1]^2 &= \text{All points in the unit square} \\
[0,1] &= \text{All points in a straight line segment of length 1}
\end{aligned}$$

Take a step back and think about what we've just discovered: since $|\mathbb{R}^3| = |[0,1]|$, we've shown that there are the same number of points in all of 3D space as there are in just a single edge of a cube!

Now you have to admit... that's pretty cool!!