Grade 6 Math Circles
February 15th, 2022
Percentages and Applications Extension

Introduction

We discussed simple interest in the lesson last week. Today, we will look at a new type of interest, but first, we will need to learn about the E in BEDMAS.

Exponents

Exponents are used to represent repeated multiplication. This is similar to how multiplication is used to represent repeated addition (e.g. $4 \times 3 = 3 + 3 + 3 + 3$).

Exponents look like a small number attached to the top right corner of another number, like the red number below. The number an exponent is attached to is called the **base**, and is represented as the blue number below.

$$4^3$$

When we have an exponent, it means that we should multiply the *base* by itself the same number of times as the exponent. For example,

$$4^3 = 4 \times 4 \times 4 = 16 \times 4 = 64$$

We usually use exponents as a way to shorten our equations. When we see an exponent and want to read it out, we would say "*base* to the power of *exponent*". For example, with our example above, we would say "4 to the power of 3".

Exercise 1

Calculate the following.

(a) 3^4

(b) 2^3

(c) 10^4

(d) $(2 + 3)^2$

(e) $(3 - 2)^8$

(f) $4 \times (1 + 1)^2 - 6$

Exercise 1 Solution

(a) $3^4 = 3 \times 3 \times 3 \times 3 = 9 \times 3 \times 3 = 27 \times 3 = 81$

(b) $2^3 = 2 \times 2 \times 2 = 4 \times 2 = 8$

(c) $10^4 = 10 \times 10 \times 10 \times 10 = 100 \times 10 \times 10 = 1000 \times 10 = 10\,000$

(d) $(2+3)^2 = (5)^2 = 5 \times 5 = 25$

(e) $(3-2)^8 = (1)^8 = 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 = 1$

(f) $4 \times (1+1)^2 - 6 = 4 \times (2)^2 - 6 = 4 \times (2 \times 2) - 6 = 4 \times 4 - 6 = 16 - 6 = 10$

Now that you know what exponents are and how to calculate them, we can introduce our new type of interest.

Compound Interest

Compound interest is a type of interest based on the original amount borrowed (or stored), the interest rate, the time borrowed (or stored), **and** the interest already earned.

We can calculate accumulated value with compound interest using the formula $A = P(1+r)^t$.

Stop and Recall

What do the letters A, P, r, and t stand for, according to last week's lesson?

Example 1

Jones borrowed \$1000 from a company for 3 years with a compound interest rate of 10%. What is the accumulated value that he will have to repay?

In this case, $P = \$1000$, $r = 0.1$, and $t = 3$. So, Jones will have to repay

$$\begin{aligned} A &= \$1000 \times (1+0.1)^3 \\ &= \$1000 \times (1.1)^3 \\ &= \$1000 \times (1.1 \times 1.1 \times 1.1) \\ &= \$1000 \times (1.331) \\ &= \$1331. \end{aligned}$$

If Jones instead borrowed the money with a simple interest rate of 10%, the accumulated value would have been $A = P(1 + rt) = \$1000 \times (1 + 0.1 \times 3) = \$1300$. Where does the extra \$31 come from?

As said above, compound interest involves interest already earned in the previous interest period, whereas simple interest only ever bases interest on the original amount.

With simple interest, every year Jones would owe an extra \$100 in interest because that's 10% of \$1000. Whereas, with compound interest, the following would happen:

- Jones borrows \$1000. At the end of that year, 10% interest is added to the \$1000 currently borrowed, so he now owes $\$1000 + \$100 = \$1100$.
- After another year, another 10% in interest is added based on his total from the end of last year. Since the last total was \$1100, the amount of interest added is equal to $\$1100 \times 0.1 = \$110$, and therefore the current total is $\$1100 + \$110 = \$1210$.
- At the end of the final year Jones borrows the money for, another 10% in interest is added based on his total from the end of last year. Since the last total was \$1210, the amount of interest added is equal to $\$1210 \times 0.1 = \$121$, and therefore the current total is $\$1210 + \$121 = \$1331$. Furthermore, since this was the final year he borrowed money for, \$1331 is also the accumulated value that he will end up repaying.

Stop and Think

If you stored money in a savings account at the bank, would you prefer to get simple interest or compound interest added to your account?

Nominal Rates

Sometimes with compound interest, the interest is calculated and added more than just one time a year. For example, interest might be calculated twice a year, four times a year, or every month in the year.

The example with Jones added interest at the end of every year, so we would say the interest was "compounded annually". We use "compounded" to refer to the calculation and adding of interest to the amount borrowed (or stored), and "annually" means "once a year". If we come across a problem where the interest is compounded more frequently, we will have to make some small changes to our formula for accumulated value.

The updated formula we would want to use is $A = P(1 + \frac{r}{m})^n$, where m is how many times the interest is compounded per year, and $n = t \times m$ represents how many times the interest will be compounded in total.

Usually, we won't be given a number for m. Instead, the following sayings might be used:

- "Compounded annually" which means $m = 1$
- "Compounded semi-annually" which means $m = 2$
- "Compounded quarterly" which means $m = 4$
- "Compounded monthly" which means $m = 12$
- "Compounded weekly" which means $m = 52$
- "Compounded daily" which means $m = 365$

Example 2

Jones borrows \$1000 for 3 years again, but this time the company's interest rate is 10% compounded quarterly. What will be the accumulated value that Jones has to repay this time?

In this case, we will use our updated formula with $P = \$1000$, $r = 0.1$, $m = 4$, and $n = 3 \times 4 = 12$. We can calculate that Jones will have to repay

$$\begin{aligned} A &= \$1000 \times (1 + 0.1 \div 4)^{12} \\ &= \$1000 \times (1 + 0.025)^{12} \\ &= \$1000 \times (1.025)^{12} \\ &= \$1000 \times 1.3448888242... \\ &= \$1344.89. \end{aligned}$$

As we can tell with this example, the more times per year we compound interest, the more interest we will have to pay (or the more money we earn).

Now, let's finish by solving some problems using what we've learned.

Exercise 2

Calculate the accumulated value for the following situations.

(a) Rahul puts \$20 500 into his savings account at the bank. If the bank offers a 6% interest rate compounded monthly, what will be the accumulated value of Rahul's account after 5 years?

(b) Luis borrows \$34 400 from a company for 18 months with a 4% interest rate compounded quarterly. What will be the accumulated value that Luis has to repay?

(c) Hannah and her sister, Miley, both put \$15 000 into a savings account. Hannah's bank account has 3% interest compounded monthly. Miley's bank account has 4% interest compounded annually. Who will earn more interest in 4 years?

Exercise 2 Solution

(a) In Rahul's case, we have that $P = \$20\,500$, $r = 0.06$, $m = 12$, and $m = 5 \times 12 = 60$. So, we can calculate that the accumulated value of Rahul's account will be

$$\begin{aligned} A &= \$20\,500 \times (1 + 0.06 \div 12)^{60} \\ &= \$20\,500 \times (1 + 0.005)^{60} \\ &= \$20\,500 \times (1.005)^{60} \\ &= \$20\,500 \times 1.3488501525... \\ &= \$27\,651.43 \end{aligned}$$

(b) In Luis's case, we have that $P = \$34\,400$, $r = 0.04$, $m = 4$, and $n = \frac{18}{12} \times 4 = 1.5 \times 4 = 6$. So, we can calculate that the accumulated value that Luis will have to repay is

$$\begin{aligned} A &= \$34\,400 \times (1 + 0.04 \div 4)^{6} \\ &= \$34\,400 \times (1 + 0.01)^{6} \\ &= \$34\,400 \times (1.01)^{6} \\ &= \$34\,400 \times 1.0615201506... \\ &= \$36\,516.29 \end{aligned}$$

(c) We can calculate the sister's accumulated values separately and then compare them to see who earned more interest.

In Hannah's case, $P = \$15\,000$, $r = 0.03$, $m = 12$, and $n = 4 \times 12 = 48$. So, Hannah's accumulated value will be

$$\begin{aligned} A &= \$15\,000 \times (1 + 0.03 \div 12)^{48} \\ &= \$15\,000 \times (1 + 0.0025)^{48} \\ &= \$15\,000 \times (1.0025)^{48} \\ &= \$15\,000 \times 1.127328021... \\ &= \$16\,909.92 \end{aligned}$$

In Miley's case, we can use the simpler formula since her interest is compounded annually with $P = \$15\,000$, $r = 0.04$, and $t = 4$. So, Miley's accumulated value will be

$$\begin{aligned} A &= \$15\,000 \times (1 + 0.04)^{4} \\ &= \$15\,000 \times (1.04)^{4} \\ &= \$15\,000 \times 1.16985856 \\ &= \$17\,547.88 \end{aligned}$$

Comparing these two values, we see that that Miley's accumulated value is more than Hannah's accumulated value. Therefore, Miley must have earned more interest because they started with the same amount of $\$15\,000$.