Grade 6 Math Circles

February 23rd, 2022

Circles

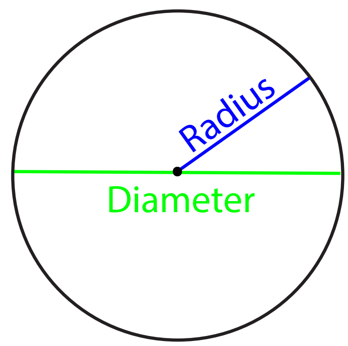
Introduction

Circular shapes are very common in everyday life, and all circles have properties that we can use to describe them and know more about them.

To begin, we define two main measurements that can be used for circles, called diameter and radius.

Diameter is the length from one side of the circle to the exact opposite side, going through the centre of the circle. We often use d to represent diameter.

Radius is the length from the edge of the circle to the centre of the circle. We often use r to represent radius. Note that $2 \times r = d$.

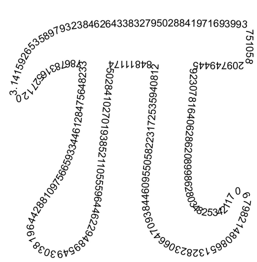


We can use these two measurements to calculate other things about circles, such as their area.

Pi

Those calculations also involve a special number that's associated with circular measurements.This special number is called **pi** (pronounced as "pie"). We use the symbol π to represent pi.

Mathematicians have defined that $\pi = 3.14159265358979323846264338327950288419716939937 51...$, where the decimal expansion goes on forever. Generally, we will just round pi to 3.14 for calculations, or alternatively we can use the π button on our calculators which uses many more decimal places.

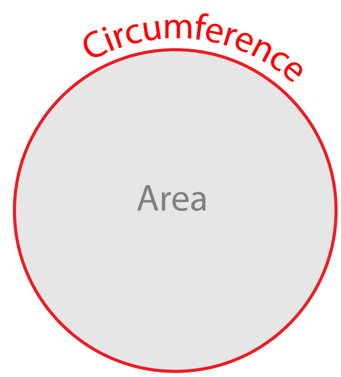


Circumference and Area

Two other measurements that circles have are called circumference and area.

Circumference is the length of the outer edge or perimeter of the circle. The formula for circumference is $C = \pi d$ or $C = 2\pi r$. Remember that d is diameter and r is radius.

Area is the measurement of the space inside the circle, and is usually measured in units to the power of 2, like cm^2 (you'll see why in the following example). The formula for area is $A = \pi r^2$.



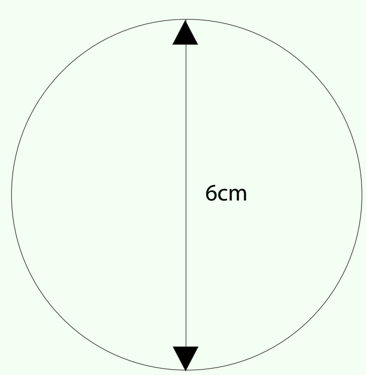
Recall that when we write numbers and variables beside each other, it implies multiplication. So, for example $C = 2\pi r$ is equivalent to $C = 2 \times \pi \times r$. Furthermore, recall that exponents represent repeated multiplication, so we know that $A = \pi r^2$ is equivalent to $A = \pi \times (r \times r)$. Finally, remember that we should always apply order of operations!

Stop and Think

Why are the two formulas for circumference equivalent? Try to come up with your own explanation before reading the explanation in the following example.

Example 1

What is the diameter, radius, circumference, and area of this circle? Round to one decimal point for each measurement.



The length portrayed as 6cm is the diameter, so $d = 6\text{cm}$, and since radius is half the diameter, we know that $r = 3\text{cm}$. Also note that since we will round to one decimal place, we only need to keep two decimal places in π to get an accurate answer. Thus we can use 3.14 instead of π. Now we can use all of these variables to calculate circumference, using either formula:

$$\begin{aligned} C &= \pi \times d \\ &= 3.14 \times 6\text{cm} \\ &= 18.84\text{cm} \end{aligned} \qquad\qquad \begin{aligned} C &= 2 \times \pi \times r \\ &= 2 \times 3.14 \times 3\text{cm} \\ &= 18.84\text{cm} \end{aligned}$$

Both formulas give the same answer because $d = 2 \times r$, as we noted earlier. Multiplying by π on both sides of the equation, $\pi \times d = \pi \times 2 \times r$. We see that these are our two formulas for circumference! We simply move the 2 to the front of the second formula because it's a mathematical construct (in other words, mathematicians agree to put numbers in front of letters when they are being multiplied together, since order of multiplication doesn't matter).

Next, we can also calculate area:

$$\begin{aligned} A &= \pi \times r^2 \\ &= 3.14 \times (3\text{cm})^2 \\ &= 3.14 \times (3\text{cm} \times 3\text{cm}) \\ &= 3.14 \times 9\text{cm}^2 \\ &= 28.26\text{cm}^2 \end{aligned}$$

Notice how when we calculated the exponent, we applied the exponent to both the number and the unit measurement. This is why we usually see area with a unit to the power of 2, which we sometimes call a unit "squared". Note that whenever we see an exponent of 2 attached to a unit, we assume it means only the unit is squared, like with 9cm^2, therefore putting the brackets around a whole measurement like we did with $(3\text{cm})^2$ is important to show that we're calculating the exponent of the number too.

We have now calculated all that the question asked for, so our concluding statement with our answers rounded to one decimal point is that the diameter of the circle is 6cm, the radius of the circle is 3cm, the circumference of the circle is 18.8cm, and the area of the circle is 28.3cm^2.

Exercise 1

Calculate the area and circumference of the following. Round to one decimal point.

(a) A circle with a radius of 2cm.

(b) A circle with a diameter of 7cm.

(c) A circle with a radius of 1m.

3D Shapes with Circles

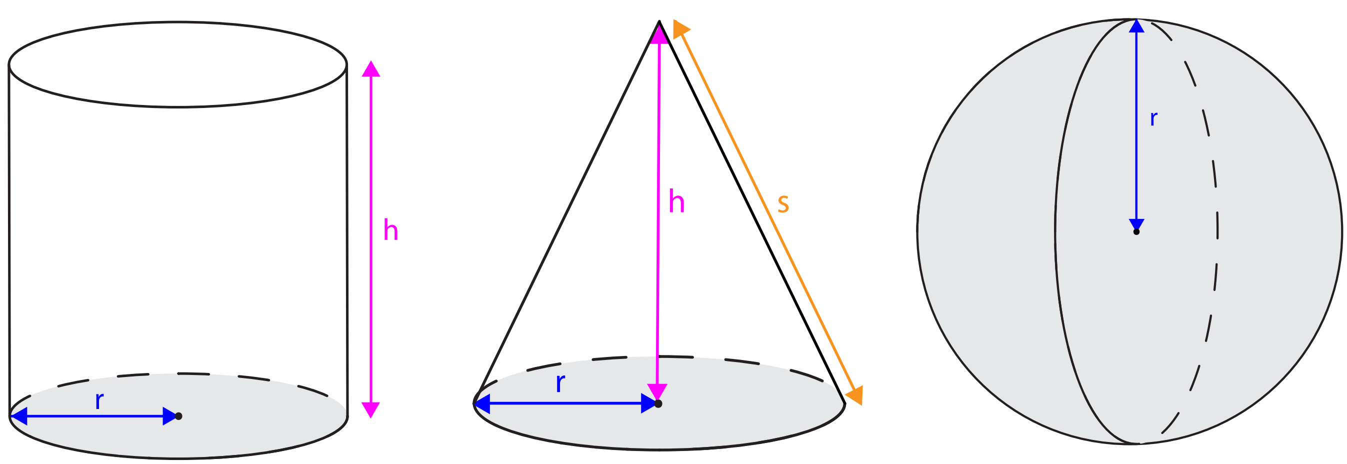
There are three common 3D shapes that include circles, called cylinders, cones, and spheres.

Cylinders are similar to rectangular prisms, which you have likely seen before in math class, except instead of a rectangle as the base, we have a circle.

Cones are similar to rectangular pyramids, which you have likely seen before in math class, except again the base is a circle.

Spheres are ball-shaped and perfectly round.

We can calculate the volume and surface area of each of these 3D shapes using the radius indicated below in each shape, but we also need to know some other measurements for a couple of the shapes. With cylinders, we also need to know the vertical height, which is represented with h. Then, with cones we will need to know the vertical height *and* the slant height, which we represent with s. These measurements are labelled in the diagrams below.



The formulas for the volume and surface area of these shapes are as follows.

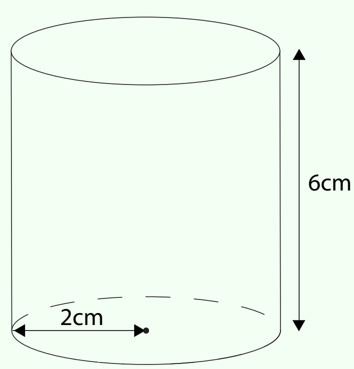
Shape	Cylinder	Cone	Sphere
Volume	$V = \pi r^2 h$	$V = \frac{1}{3}\pi r^2 h$	$V = \frac{4}{3}\pi r^3$
Surface Area	$SA = 2\pi r^2 + 2\pi rh$	$SA = \pi r^2 + \pi rs$	$SA = 4\pi r^2$

At first glance, these seem like very complicated formulas because of all the variables, but breaking it down, it's just multiplications and some additions. Furthermore, when we understand where these formulas come from, it can make them easier to understand and use.

Starting with cylinders, the volume formula can be seen as consisting of two main parts. First, there is the πr^2 part. We can recognize that this is the formula for area of a circle, and this calculation will give us the area of the circular base! Then, we have the height, h. So, describing the formula in words, we can see that we are multiplying the area of the base by the height of the cylinder. It makes sense as to why this would equal the volume, since volume is like the 3D version of area. This is similar to calculating $V = (\textit{length} \times \textit{width} \times \textit{height})$ for rectangular prisms! Next, we can look at the surface area formula and see two main parts again. First, there is the $2\pi r^2$ part, where we can again recognize the formula for the area of a circle, and this time it's multiplied by two. This makes sense because we have two circles at each end of the cylinder that will contribute to surface area. Next, there is the $2\pi rh$ part, where we can recognize that $2\pi r$ is one of the formulas for circumference, and then multiplying that by h is like stacking up the outer edge of the circle to give us the surface area of the side of the cylinder. These three surfaces, the two circle bases and the side, make up all the surface area of the whole cylinder!

Example 2

What is the volume and surface area of the cylinder below? Round to one decimal point.



We have that $r = 2\text{cm}$ and $h = 6\text{cm}$. We can also use 3.14 instead of π since we only need one decimal point of accuracy. Now, using our formulas for volume and surface area, we get that

$$\begin{aligned} V &= 3.14 \times (2\text{cm})^2 \times 6\text{cm} \\ &= 3.14 \times (2\text{cm} \times 2\text{cm}) \times 6\text{cm} \\ &= 3.14 \times 4\text{cm}^2 \times 6\text{cm} \\ &= 75.36\text{cm}^3 \end{aligned}$$

$$\begin{aligned} SA &= 2 \times 3.14 \times (2\text{cm})^2 + 2 \times 3.14 \times 2\text{cm} \times 6\text{cm} \\ &= 2 \times 3.14 \times (2\text{cm} \times 2\text{cm}) + 2 \times 3.14 \times 2\text{cm} \times 6\text{cm} \\ &= 2 \times 3.14 \times 4\text{cm}^2 + 2 \times 3.14 \times 2\text{cm} \times 6\text{cm} \\ &= 25.12\text{cm}^2 + 75.36\text{cm}^2 \\ &= 100.48\text{cm}^2 \end{aligned}$$

Therefore, rounding to one decimal point, the volume of the cylinder is 75.4cm^3 and the surface area of the cylinder is 100.5cm^2.

Exercise 2

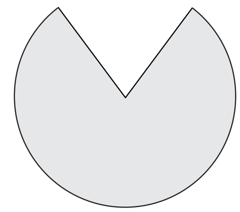
Calculate the volume and surface area of the following cylinders. Round to one decimal point.

(a) A cylinder with a radius of 4m and a height of 3m.

(b) A cylinder with a radius of 6cm and a height of 2cm.

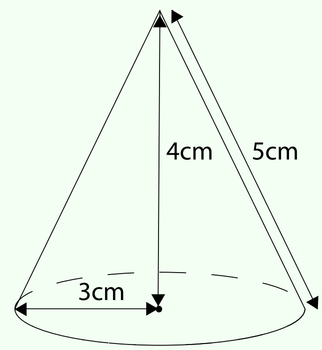
Next, we can break down the formulas for cones. When we look at the formula for volume, we see that it's the same as the formula for a cylinder except multiplied by $\frac{1}{3}$. This means that when we have a cylinder and a cone with the same radius and vertical height, the volume of the cylinder is $\frac{1}{3}$ of the volume of the cone. If you want to see this visually, watch this YouTube video where they demonstrate this property: https://youtu.be/0ZACAU4SGyM. Now, looking at the formula for surface area, there's two main parts, like with the cylinder. The first part is πr^2, which we know is the area of a circle, and is therefore the contribution from the circular base of the cone. Then, there's the πrs part, which must represent the side of the cone, where we have the height of the slant multiplied by πr.

When we look at sectors of circles later, this will make sense because laying the side of the cone flat looks like a circle with a triangle cut out of it, similar to the cutout to the right.



Example 3

What is the volume and surface area of this cone? Round to three decimal points.



We have that $r = 3\text{cm}$, $h = 4\text{cm}$, and $s = 5\text{cm}$. Also, we can use 3.1416 instead of π since we only need three decimal points of accuracy. Now, using our formulas for volume and surface area of a cone, we get that

$$\begin{aligned} V &= \frac{1}{3} \times 3.1416 \times (3\text{cm})^2 \times 4\text{cm} \\ &= \frac{1}{3} \times 3.1416 \times (3\text{cm} \times 3\text{cm}) \times 4\text{cm} \\ &= \frac{1}{3} \times 3.1416 \times 9\text{cm}^2 \times 4\text{cm} \\ &= 37.6992\text{cm}^3 \end{aligned}$$

$$\begin{aligned} SA &= 3.1416 \times (3\text{cm})^2 + 3.1416 \times 3\text{cm} \times 5\text{cm} \\ &= 3.1416 \times 9\text{cm}^2 + 3.1416 \times 3\text{cm} \times 5\text{cm} \\ &= 28.2744\text{cm}^2 + 47.124\text{cm}^2 \\ &= 75.3984\text{cm}^2 \end{aligned}$$

Therefore, rounding to three decimal points, our final answer is that the volume of the cone is 37.699cm^3 and the surface area of the cone is 75.398cm^2.

Exercise 3

Calculate the volume and surface area of a cone with a radius of 6cm, a vertical height of 8cm, and a slant height of 10cm. Round to one decimal point.

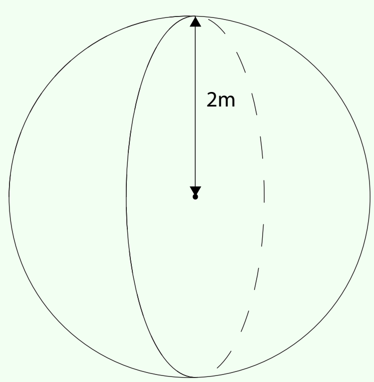
To finish this section, we can now break down the formulas for spheres. Starting with the volume formula, an interesting observation can be made. If we have a sphere and a cylinder with the same radius, and the height of the cylinder is equal to $2r$, then the volume of the sphere will be $\frac{2}{3}$ of the volume of the cylinder. If you want to see this visually, watch this YouTube video where they demonstrate this property: https://youtu.be/h4j8l3p22e8. Algebraically, we can see how we got the formula for spheres from the formula for cylinders now that we know the property just described:

$$\begin{aligned} \text{Volume of sphere} &= \frac{2}{3} \times \pi \times r^2 \times h \\ &= \frac{2}{3} \times \pi \times r^2 \times 2r \quad (\text{since } h = 2r) \\ &= 2 \times \frac{2}{3} \times \pi \times r^2 \times r \quad (\text{since we can rearrange multiplication}) \\ &= \frac{4}{3} \times \pi \times r^3 \end{aligned}$$

Now, looking at the volume for surface area, we can see that it's 4 multiplied by the formula for the area of a circle. So, the surface area of a sphere is 4 times the area of the circle it's made of. This can be explained visually, if you watch the following YouTube video: https://youtu.be/jaL8Kuv6YHo.

Example 4

What is the volume and surface area of the sphere below? Round to one decimal point.



We have that $r = 2\text{cm}$ and we can use 3.14 instead of π since we only need one decimal point of accuracy. So, using the formulas for volume and surface area of spheres, we get that

$$\begin{aligned} V &= \frac{4}{3} \times 3.14 \times (2\text{cm})^3 \\ &= \frac{4}{3} \times 3.14 \times (2\text{cm} \times 2\text{cm} \times 2\text{cm}) \\ &= \frac{4}{3} \times 3.14 \times 8\text{cm}^3 \\ &= 33.4933333...\text{cm}^3 \end{aligned}$$

$$\begin{aligned} SA &= 4 \times 3.14 \times (2\text{cm})^2 \\ &= 4 \times 3.14 \times (2\text{cm} \times 2\text{cm}) \\ &= 4 \times 3.14 \times 4\text{cm}^2 \\ &= 50.24\text{cm}^2 \end{aligned}$$

Therefore, rounded to one decimal point, the volume of the sphere is 33.5cm^3 and the surface area of the sphere is 50.2cm^2.

Exercise 4

Calculate the volume and surface area of the following spheres. Round to one decimal point.

(a) A sphere with a radius of 1m.

(b) A sphere with a diameter of 6cm.

Conclusion

As you've seen, there are many different aspects to circles, including measurements and 3D shapes. There are even more properties of circles that you might learn in the future.

As a fun activity, try to go around your home and find objects that you can now find the volume and surface area of (like a can of soup)!