Grade 6 Math Circles
March 9, 2022
The A, B, Γ's of Math

The Greek Alphabet

What's your favourite thing about Greek culture? Maybe you've vacationed in Greece and were able to see the picturesque views of their cities. Maybe you love Ancient Greek mythology and enjoy reading about their magnificent gods. Maybe you like eating Greek salad. Or just maybe, you can read or speak the Greek language.

Spoken in Greece, Cyprus, Albania, and many more countries, the Greek language is a very old and well documented language of human history, dating back thousands of years. Their writing system, known as the Greek Alphabet, is given below.

Aα	Bβ	$\Gamma\gamma$	$\Delta\delta$	Eϵ	Zζ	Hη	$\Theta\theta$	Iι	Kκ	$\Lambda\lambda$	Mμ
alpha	beta	gamma	delta	epsilon	zeta	eta	theta	iota	kappa	lambda	mu
Nν	$\Xi\xi$	Oo	$\Pi\pi$	Pρ	$\Sigma\sigma$	Tτ	$\Upsilon\upsilon$	$\Phi\phi$	Xχ	$\Psi\psi$	$\Omega\omega$
nu	xi	omicron	pi	rho	sigma	tau	upsilon	phi	chi	psi	omega

Like the modern English alphabet, each Greek letter has both an uppercase and a lowercase form. Additionally, the Greek alphabet has a distinction between vowels and consonants. There exist both similarities and major differences between the letters of the Greek and modern English alphabet.

Exercise A

Try to spell your name using Greek letters.

Each Greek letter has at least one use in the modern age. Many of them are used in mathematics, but many letters are used all throughout different branches of science. Instead of going into each letter in "alphabetical" order, we will look into different pockets of mathematics, exploring how the letters are used and what they can possibly mean.

Stop and Think

Which letters in the Greek Alphabet are familiar? Which names of the letters have you heard before and where have you heard them?

Mathematical Variables and Constants

A **variable** is a symbol for an unknown value. Often times variables represent numbers, and we often choose lowercase letters like a, b, c, x, y, etc. as the symbols.

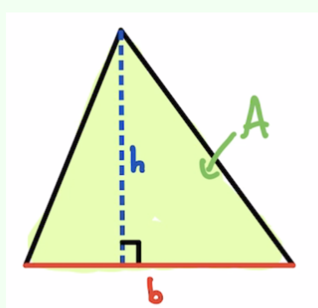
Example A

Consider the formula for the area of a triangle:

$$A = \frac{b \times h}{2}$$

We use this formula for any triangle, which is why we need variables in the equation; the side lengths, angles, and areas are not necessarily the same for every triangle.

- b is a variable that represents the base of a triangle.
- h is a variable that represents the height of a triangle.
- A is also a variable that represents an area.



Furthermore, we could assign values to each of these variables. For example, letting $b = 4$ units and $h = 3$ units, we have an area $A = 6$ square units.

Note that in the previous example, 2 (or $\frac{1}{2}$) is not a variable. The number 2 is a **constant**, as the value of 2 is always the same.

Exercise B

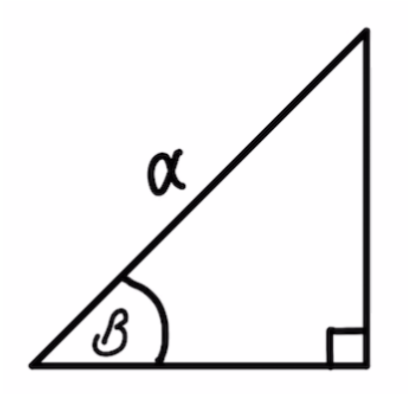
Determine what the variables are in each mathematical equation. If you are familiar with the equation, determine what each variable represents.

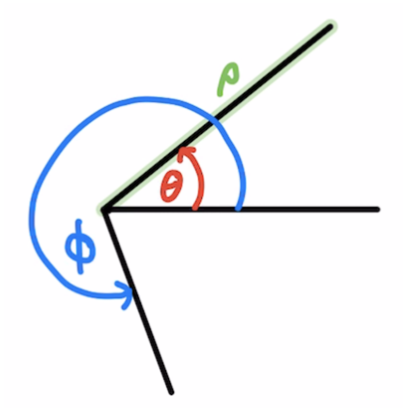
(α) $C = 2\pi r$ (circumference of a circle)

(β) $a^2 + b^2 = c^2$ (Pythagorean Theorem)

Trigonometric Variables

In trigonometry, the study of triangles, there are a few options for variables that we use. The letters α (**alpha**) and β (**beta**) are used to represent both unknown side lengths or unknown angles in a triangle. In higher levels of trigonometry and calculus, we might use ρ (**rho**) to represent lengths and use θ (**theta**) or ϕ (**phi**) to represent angles.





In both levels of trigonometry we typically aim to solve for values to assign to each variable. This is why you might also see these variables referred to as "unknowns"; we want to find the unknown side length or the unknown angle.

Constants and Exact Values

If there's any letter in the Greek alphabet that you've heard of, it's likely π (**pi**). In fact, it was a major concept in last week's lesson on Circles.

When we say that π "never ends", we mean it has an infinite amount of decimal places; it's an irrational number. What the **constant** π allows us to do is provide **exact values**. This means we do not have to approximate values to a certain number of decimal places.

Example B

Recall that the area of a circle is given by $A = \pi r^2$, where the r is the radius of the circle.

For a circle with radius $r = 2$:

- An approximate area of of this circle is $A = (3.14)2^2 \approx 12.57$ square units.
- The **exact** area of this circle is $A = (\pi)2^2 = \pi(4) = 4\pi$ square units.

Here, π is a number. Consequently 4π is also a number.

Using constant symbols to get exact values is useful because it keeps our math looking clean and none of our decimal values are lost at any point in the process.

Exercise Γ

We say that the constant τ (**tau**) has the following relationship to π.

$$\tau = 2\pi$$

(α) Find the approximate value of τ to 2 decimal places.

(β) Find an approximate circumference of a circle with radius $r = 7$. Round to 2 decimal places.

(γ) Find the exact value of the circumference of a circle with radius $r = 28$.

Statistics

In statistics and data analysis, we sometimes use the Greek letter μ (**mu**) to represent the **mean** (or average) of a data set.

Exercise Δ

Amogh rolls a dice 9 times and records the result. Calculate the mean, μ, of his data:

$$\{2, 4, 6, 5, 5, 3, 2, 4, 5\}$$

Recall that we calculate the mean by taking the sum of the numbers in the list and dividing by how many numbers there are.

We also use the letter σ (**sigma**) to represent the **standard deviation** of a data set and take σ^2 to be the **variance**. These are two other characteristics of data that allow us to make predictions and make statistical conclusions on data.

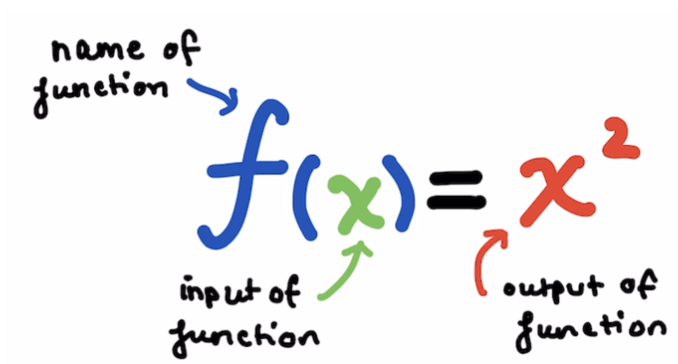
Number Theory Functions

So far, we have seen various Greek letters used as variables in different areas of mathematics, including geometry and statistics. I now want to turn our attention to Number Theory, a branch of pure mathematics that focuses on integers (or whole numbers) and **integer functions**.

Function Notation

An integer function, $f(x)$, usually has three components:

- a name, usually a letter like f, but could be anything,
- an input, denoted by a variable x which is a placeholder for an integer, and
- an output, usually given in terms of x, and ultimately an integer value.



Example Γ

Let's make a function f that takes an integer input and then outputs the double of that integer. We'll let x be the variable input. We want the output to be $x \times 2$ or $2x$. We define

$$f(x) = 2x$$

Let's now give our function inputs $x = 3$, $x = 10$, and $x = 45$. We have

$$\begin{aligned} f(3) &= 2(3) \\ &= 6 \end{aligned} \qquad \begin{aligned} f(10) &= 2(10) \\ &= 20 \end{aligned} \qquad \begin{aligned} f(45) &= 2(45) \\ &= 90 \end{aligned}$$

Exercise E

Given that $f(x) = 3x$ and $g(x) = x + 6$ calculate the following.

(α) $f(5)$

(β) $g(9)$

(γ) $g(104) - f(20)$

We'll see that functions are not always defined using standard arithmetic. The next few functions use words to describe their intended output, starting with this next function with a somewhat familiar name.

Divisors function

Hopefully we know what a **divisor** is. If not, please see this lesson on Divisibility from the Fall if you would like a review.

Divisors function, $\sigma(x)$

Define the **divisors function**, $\sigma(x)$ as the sum of the divisors of integer x.

Example Δ

The divisors of 20 are 1, 2, 4, 5, 10, and 20. Therefore

$$\begin{aligned}\sigma(20) &= 1+2+4+5+10+20\\ &= 42\end{aligned}$$

Exercise Z

Calculate the following.

(α) $\sigma(32)$

(β) $\sigma(37)$

Prime-Counting function

Recall that a **prime number** is a positive integer whose only divisors are 1 and itself (e.g. 2, 3, 5, 7, etc.). Then, a positive integer with more than 2 divisors is a **composite number** (e.g. 4, 6, 8, 9, etc.). We say that 0 and 1 are not prime and not composite.

Prime-counting function, $\pi(x)$

Define $\pi(x)$ to be the **prime-counting function**. The function $\pi(x)$ is the number of prime numbers less than or equal to integer x.

Example E

The prime numbers less than or equal to 20 are: 2, 3, 5, 7, 11, 13, 17, and 19. Therefore

$$\pi(20) = 8$$

It's as simple as it sounds, once you get over the fact that the function $\pi(x)$ is not at all the same as the constant $\pi \approx 3.14$. They just happen to share the same name.

Exercise H

What is $\pi(19)$? What about $\pi(22)$?

Predict the value of $\pi(100)$ given that $\pi(90) = 24$.

Sigma Summation Notation

Remember how Greek letters have both uppercase and lowercase variations? Well, in math they have different meanings. We saw the letter σ (sigma) used as a variable in statistics, and in the previous section, we saw $\sigma(x)$ as the divisors function. Now we have capital Σ (**Sigma**) which is an *operator* that means **"summation"** or simply **"sum"**.

First off, we interpret $\sum n$ as the "sum of n". What is n? This is better shown with an example.

Example Z

Consider

$$\sum_{n=1}^{5} n$$

- The $n = 1$ **below** the Σ means that the values of n start at 1.
- The 5 **above** the Σ means that the values of n end at 5.
- n is what comes after Sigma and what we are summing up.

Therefore n takes on the values 1 to 5. The expression $\sum\limits_{n=1}^{5} n$ is thus the sum of values 1 to 5. Thus

$$\sum_{n=1}^{5} n = 1 + 2 + 3 + 4 + 5 = 15$$

Exercise I

Evaluate each sum.

(α) $\sum\limits_{n=0}^{7} n$

(β) $\sum\limits_{i=18}^{21} i$

As of right now, this seems like a very inconvenient way to add numbers. You're probably right, but Sigma summations are useful when there's too many things to add up, including when we sum up numbers up to infinity. Below is sum of all positive whole numbers shown using summation notation.

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + \cdots = \infty$$

We can also sum up expressions that aren't just single numbers.

Example H

Evaluate each sum.

(α) $\sum_{i=1}^{6}(i+3)$

(β) $\sum_{n=1}^{8} 4n$

Solution:

(α) For each value between 1 and 6 we add a 3. Thus we take a sum of sums.

$$\begin{aligned}\sum_{i=1}^{6}(i+3) &= (1+3) + (2+3) + (3+3) + (4+3) + (5+3) + (6+3)\\ &= 4+5+6+7+8+9\\ &= 39\end{aligned}$$

(β) Instead of values between 1 and 8 we are summing up the first 8 multiples of 4. We take a sum of products.

$$\begin{aligned}\sum_{n=1}^{8} 4n &= 4(1) + 4(2) + 4(3) + 4(4) + 4(5) + 4(5) + 4(6) + 4(7) + 4(8)\\ &= 4+8+12+16+20+24+28+32\\ &= 144\end{aligned}$$

Exercise Θ

Write out the expanded form of each summation and then evaluate the sum.

(α) $\sum_{n=1}^{10}(2n-1)$

(β) $\sum_{n=1}^{4}\left(\frac{1}{n+2}\right)$

Hopefully you had *sum* fun learning this new notation. In the problem set we will look at different manipulations of summations that include calculation shortcuts and some practical applications.

We can also describe some of the functions from earlier using summation notation. Below is an expression of $\sigma(20)$ using summation notation.

$$\begin{aligned}\sigma(20) &= 1+2+4+5+10+20\\ &= \sum_{n|20} n\end{aligned}$$

Conclusion

As a conclusion to our lesson, and a hint about our next one, here is a message for you decipher now that you know a bit more about the Greek Alphabet and the uses of its letters.

$$\text{“}i \quad 2^3 \quad \Sigma \quad \pi\text{”}$$