Grade 7/8 Math Circles
February 23, 2022
Boolean Algebra - Solutions

Exercise Solutions

Exercise 1

Consider the question "Do you have a pet cat?".

Give a Boolean response to this question.

Exercise 1 Solution

Answers may vary.

- If you have a pet cat, then possible responses are "Yes" or "True".
- If you do not have a pet cat, then possible responses are "No" or "False".

Exercise 2

Consider the question "Do you like apples **or** bananas?"

Suppose that you like apples and you like bananas. Would you respond with "Yes" or "No"?

Exercise 2 Solution

Answers may vary! Assuming you like apples and you like bananas, you could answer:

- "Yes", since you like apples, but also happen to like bananas. You like *at least* one of them.
- "No", since you like both and not just one of the two fruits.

You're answer depends on how you interpret the question, and more specifically how you interpret the meaning of "or". You would say "Yes" if you're using the definition of OR given in the lesson. You would say "No" if using XOR.

Alternatively you interpret the question without our Boolean subtext and say something like "I like both apples and bananas!". This is fine and it perfectly communicates how you feel about apples and bananas without ambiguity.

Exercise 3

Evaluate the following Boolean expressions. You may use the truth table as a reference if needed.

(a) $\neg 0$

(b) $0 \wedge 1$

(c) $1 \wedge 1$

(d) $0 \vee 0$

(e) $1 \vee 1$

(f) $1 \oplus 0$

Exercise 3 Solution

(a) $\neg 0 = 1$

(b) $0 \wedge 1 = 0$

(c) $1 \wedge 1 = 1$

(d) $0 \vee 0 = 0$

(e) $1 \vee 1 = 1$

(f) $1 \oplus 0 = 1$

Exercise 4

Evaluate the following in respect to the order of operations where $A = 1$, $B = 1$, and $C = 0$.

(a) $A \vee C \wedge B$

(b) $(A \oplus C) \wedge \neg B$

Exercise 4 Solution

Start with substitutions $A = 1$, $B = 1$, and $C = 0$.

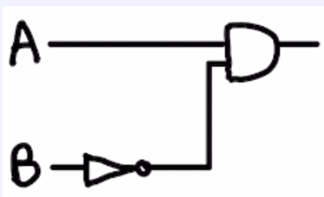
(a) $A \vee C \wedge B$

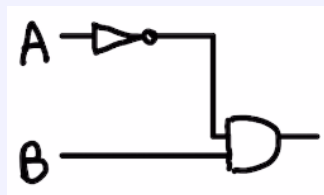
$$\begin{aligned} 1 \vee 0 \wedge 1 &= 1 \vee (0) && \text{(AND before OR)} \\ &= 1 \end{aligned}$$

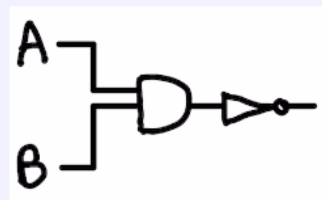
(b) $(A \oplus C) \wedge \neg B$

$$
\begin{aligned}
(1 \oplus 0) \wedge \neg 1 &= (1) \wedge \neg 1 && \text{(XOR in Brackets)} \\
&= 1 \wedge (0) && \text{(NOT before AND)} \\
&= 0
\end{aligned}
$$

Exercise 5
Determine values for inputs A and B such that the output of the circuit is 1.

(a) 

(b) 

(c) 

Exercise 5 Solution

(a) $A = 1$, $B = 0$

(b) $A = 0$, $B = 1$

(c) $A = 0$, $B = 0$ or $A = 1$, $B = 0$ or $A = 0$, $B = 1$

Problem Set Solutions

1. Recall that everyday logic does not translate perfectly with Boolean logic.
   (a) Respond to the following questions with an everyday response and then with a Boolean response if possible.
      i. Will you not be at the birthday party?
      ii. Would you like ice cream after lunch or after dinner?
      iii. What is the temperature outside?
   (b) Why does Boolean logic not always apply in real life?

Solution: Answers may vary.

(a) There are no incorrect answers, but here are some possible responses.

i. "I will not"; "True"

ii. "After lunch"; "Yes"

iii. "It's 3 degrees Celsius"; (likely you got nothing here)

(b) Not all things in life can be categorized into one of two things. Even then, Boolean logic does not necessarily provide useful answers (e.g. replying "Yes" to an X or Y type of question). Also Booleans don't have units; you could respond to the temperature question with "0" but that's not necessarily 0 degrees.

2. Evaluate each boolean expression. Keep in mind the order of operations.

(a) $1 \vee 0$

(b) $0 \oplus 1$

(c) $0 \vee 1 \wedge 0$

(d) $\neg(1 \wedge 0)$

(e) $\neg 0 \wedge 1$

(f) $\neg 0 \oplus \neg(1 \wedge 0)$

(g) $1 \wedge 0 \vee 1 \oplus 0 \vee \neg 1$

(h) $1 \wedge (0 \vee 1) \oplus \neg(0 \vee \neg 1)$

(i) $0 \vee 0 \vee 0 \vee 0 \vee 1 \vee 1 \vee 1$

(j) $1 \wedge 1 \wedge 1 \wedge 0 \wedge 0$

Solution:

(a) $1 \vee 0 = 1$

(b) $0 \oplus 1 = 1$

(c) $0 \vee 1 \wedge 0 = 0 \vee 0 = 0$

(d) $\neg(1 \wedge 0) = \neg(0) = 1$

(e) $\neg 0 \wedge 1 = 1 \wedge 1 = 1$

(f) $\neg 0 \oplus \neg(1 \wedge 0) = \neg 0 \oplus \neg(0) = 1 \oplus 1 = 0$

(g) $1 \wedge 0 \vee 1 \oplus 0 \vee \neg 1 = 1 \wedge 0 \vee 1 \oplus 0 \vee 0 = 0 \vee 1 \vee 0 = 1$

(h) $1 \wedge (0 \vee 1) \oplus \neg(0 \vee \neg 1) = 1 \wedge 1 \oplus \neg(0) = 1 \oplus 1 = 0$

(i) $0 \vee 0 \vee 0 \vee 0 \vee 1 \vee 1 \vee 1 = 1$

(j) $1 \wedge 1 \wedge 1 \wedge 0 \wedge 0 = 0$

3. Consider the following expressions.

   i. $\neg\neg B$
   ii. $B \vee B$
   iii. $B \wedge B$
   iv. $B \vee \neg B$
   v. $B \wedge \neg B$

   (a) Evaluate each of the expressions when $B = 0$.
   (b) Evaluate each of the expressions when $B = 1$.
   (c) Based on your answers to parts (a) and (b), simplify the following expressions for any Boolean value of A.

      i. $\neg\neg A$
      ii. $A \vee A$
      iii. $A \wedge A$
      iv. $A \vee \neg A$
      v. $A \wedge \neg A$

   (d) Simplify the following expressions of A and B. Your findings from part (c) might be helpful.

      i. $(A \vee A \vee B \vee B)$
      ii. $\neg\neg((A \wedge B) \wedge (A \wedge B))$
      iii. $(A \wedge \neg\neg B) \vee \neg\neg\neg(A \wedge B)$

Solution:

(a)
   i. $\neg\neg 0 = \neg 1 = 0$
   ii. $0 \vee 0 = 0$
   iii. $0 \wedge 0 = 0$
   iv. $0 \vee \neg 0 = 0 \vee 1 = 1$
   v. $0 \wedge \neg 0 = 0 \wedge 1 = 0$

(b)
   i. $\neg\neg 1 = \neg 0 = 1$
   ii. $1 \vee 1 = 1$
   iii. $1 \wedge 1 = 1$
   iv. $1 \vee \neg 1 = 1 \vee 0 = 1$
   v. $1 \wedge \neg 1 = 1 \wedge 0 = 0$

(c)
   i. $\neg\neg A = A$
   ii. $A \vee A = A$
   iii. $A \wedge A = A$
   iv. $A \vee \neg A = 1$
   v. $A \wedge \neg A = 0$

(d)
   i. $(A \vee A \vee B \vee B) = (A \vee B)$

ii. $\neg\neg((A \wedge B) \wedge (A \wedge B)) = \neg\neg(A \wedge B) = (A \wedge B)$

iii. $(A \wedge \neg\neg B) \vee \neg\neg\neg(A \wedge B) = (A \wedge B) \vee \neg(A \wedge B) = 1$

4. We will now introduce 3 more negation operators to complete our language of boolean algebra. **NAND** ($|$), **NOR** ($\downarrow$), and **XNOR** ($\odot$).

(a) Complete the truth table provided given that

- A NAND $B = A|B = \neg(A \wedge B)$
- A NOR $B = A \downarrow B = \neg(A \vee B)$
- A XNOR $B = A \odot B = \neg(A \oplus B)$

A	B	$\neg A$	$A \wedge B$	$A\|B$	$A \vee B$	$A \downarrow B$	$A \oplus B$	$A \odot B$
0	0	1	0		0		0	
0	1	1	0		1		1	
1	0	0	0		1		1	
1	1	0	1		1		0	

(b) Rewrite the following expressions for A, B, and C using the new negation operators.

i. $A \vee \neg(B \wedge C)$

ii. $\neg(A \oplus C) \vee \neg(A \vee C)$

iii. $(\neg A \vee B) \oplus \neg(A \wedge C)$

(c) Let us expand our Order of Operations to include NAND, NOR, and XNOR. (They have equal precedence as their unnegated counterparts).

Brackets $\rightarrow$ NOT $\rightarrow$ AND $=$ NAND $\rightarrow$ XOR $=$ XNOR $\rightarrow$ OR $=$ NOR

Evaluate the expressions in part (b) where $A = 0, B = 1, C = 0$.

Solution:

(a)

A	B	$\neg A$	$A \wedge B$	$A\|B$	$A \vee B$	$A \downarrow B$	$A \oplus B$	$A \odot B$
0	0	1	0	1	0	1	0	1
0	1	1	0	1	1	0	1	0
1	0	0	0	1	1	0	1	0
1	1	0	1	0	1	0	0	1

(b) i. $A \vee \neg(B \wedge C) = A \vee (B|C)$

ii. $\neg(A \oplus C) \wedge \neg(A \vee C) = (A \odot C) \wedge (A \downarrow C)$

iii. $(\neg A \vee B) \oplus \neg(A \wedge C) = (\neg A \vee B) \oplus (A|C)$

(c) i.

$$\begin{aligned} A \vee (B|C) &= (0) \vee (1|0) \\ &= 0 \vee (1) \\ &= 1 \end{aligned}$$

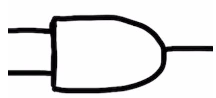
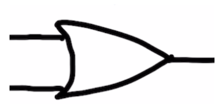
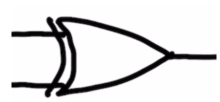
ii.

$$\begin{aligned} (A \odot C) \wedge (A \downarrow C) &= ((0) \odot (0)) \wedge ((0) \downarrow (0)) \\ &= (1) \wedge (1) \\ &= 1 \end{aligned}$$

iii.

$$\begin{aligned} (\neg A \vee B) \oplus (A|C) &= (\neg(0) \vee (1)) \oplus ((0)|(0)) \\ &= (1 \vee 1) \oplus (1) \\ &= (1) \oplus 1 \\ &= 0 \end{aligned}$$

5. Determine a boolean expression that corresponds to each circuit diagram. For reference, here are the symbols again.

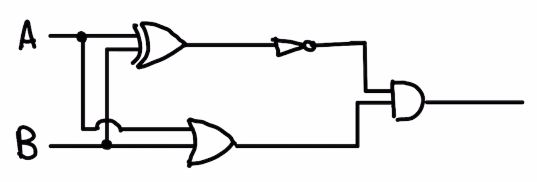
NOT ($\neg$)	**AND** ($\wedge$)	**OR** ($\vee$)	**XOR** ($\oplus$)
			

Some tips:

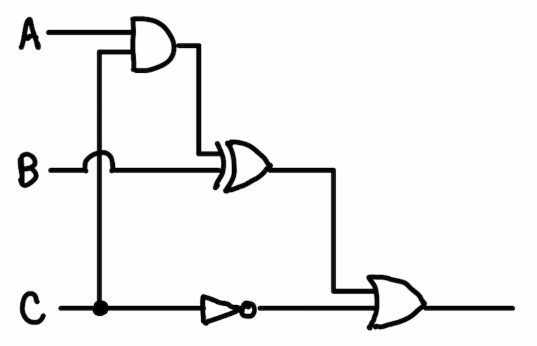
- Read the diagram from left to right (input to output).
- Label the output of every logic gate symbol as you go along.

- The little humps in the circuit are drawn to show one wire crossing over a different wire. They will not change the input, nor are they an operator.
- When coming up with an expression, you could use brackets at every step to avoid issues with precedence.

(a)

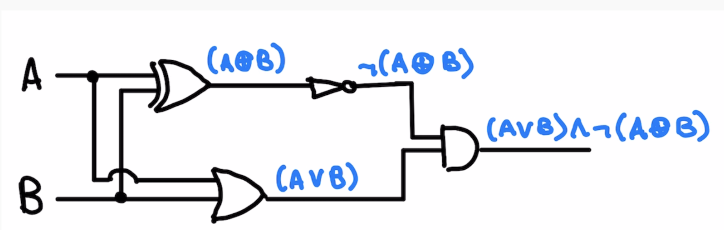


(b)



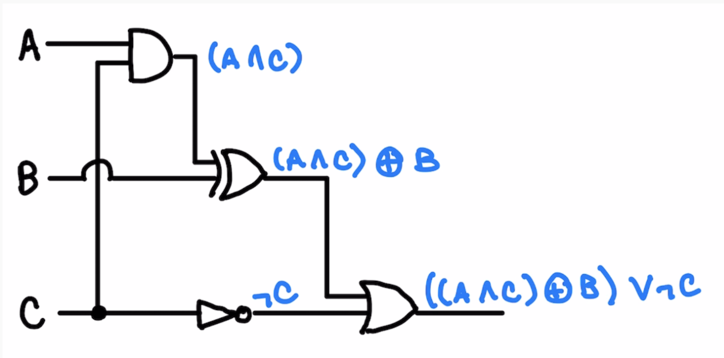
Solution:

(a) Labeling each output gives



So a corresponding expression is $(A \vee B) \wedge \neg(A \oplus B)$

(b) Labelling each output gives.



So a corresponding expression is $((A \wedge C) \oplus B) \vee \neg C$.

6. In this question you will be drawing and interpreting circuit diagrams.

   (a) Create a circuit diagram using Logic Gates for each boolean expression.

      i. $A \vee B \wedge C$

      ii. $A \wedge \neg A$

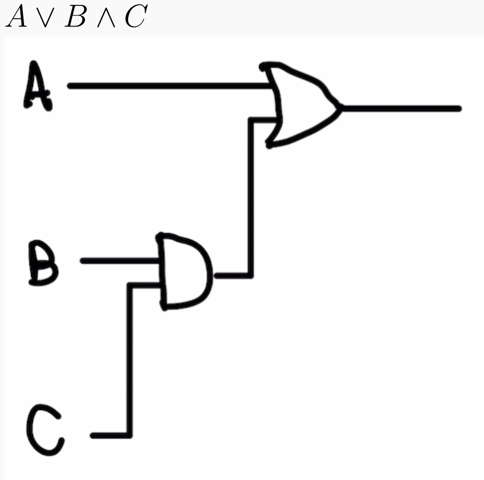
      iii. $((A \wedge B) \vee (B \wedge C)) \vee (A \wedge C)$

   (b) For each of the expressions/circuit diagrams, determine if there are possible input values for A, B, and C such that the output of the expression/circuit is 1.

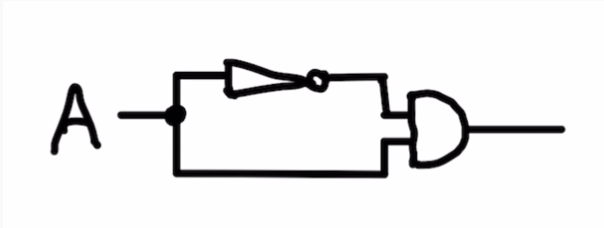
Solution:

(a) Diagrams may vary by general shape, but make sure that the inputs and outputs match up.

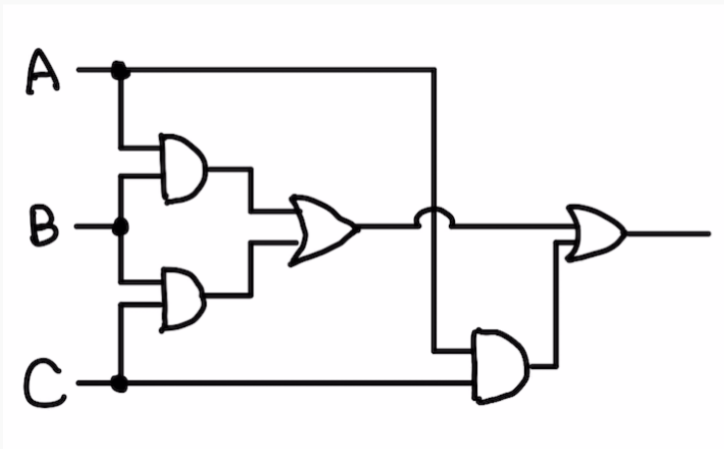
   i. $A \vee B \wedge C$



ii. $A \wedge \neg A$



iii. $((A \wedge B) \vee (B \wedge C)) \vee (A \wedge C)$



(b) Answers may vary.

i. One possible solution is $A = 1, B = 0, C = 0$. Another is $A = 0, B = 1, C = 1$.

ii. There is no possible value of A such that the output is 1.

iii. The possible solutions require at least 2 of the inputs to have a value of 1, such as $A = 1, B = 1, C = 0$.

7. Draw a circuit diagram such that the circuit output is 1 for the given inputs and required logic gates. (Hint: For some, it might be helpful to create an expression first. For others, it might be easier to go straight to the diagram.)

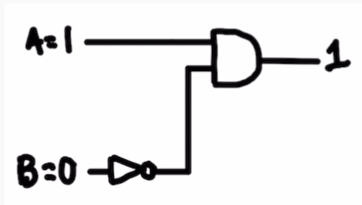
(a) With inputs $A = 1$ and $B = 0$, use 1 NOT gate and 1 AND gate.

(b) With inputs $A = 0$ and $B = 0$, use 1 OR gate and 2 NOT gates.

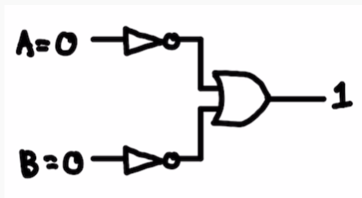
(c) With inputs $A = 0$ and $B = 1$, use 1 OR gate and 1 AND gate.

Solution: As usual, circuit diagrams may vary by shape.

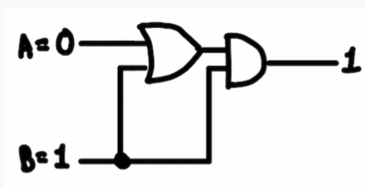
(a) This circuit has expression $A \wedge \neg B$



(b) This circuit has expression $\neg A \vee \neg B$



(c) This circuit has expression $(A \vee B) \wedge B$



8. The light-bulb in my kitchen responds to two light switches. Flipping either switch Up/Down will turn the light On/Off depending on if the light was Off/On before the flip.

   (a) In the context of my kitchen lighting, what could be represented with boolean variables?

   (b) When one switch is flipped Up and one switch is flipped Down, the light is On; otherwise, the light is Off. Create a boolean expression that represents the state of the light being On or Off.

   If you have a similar lighting situation at home, I recommend experimenting with all possible combinations. If not, how else could you test all possible switch-flipping combinations?

   (c) An electrical engineer insists on installing one more light switch for a total for three switches for one light-bulb. As before, flipping any of the switches in either direction will change the light from On to Off or from Off to On.

The light On when the three switches are flipped Up. Is it possible for the light to be On when all three switches are flipped Down?

(d) **Challenge:** A graduating class of 500 electrical engineers all have the same idea and each install another light switch for my kitchen light (503 switches in total for 1 light-bulb). As before, flipping any of the switches in either direction will change the light from On to Off or from Off to On. Assume that this is possible and that my kitchen does not catch on fire.

The light is On when all 503 switches are flipped Up. Is it possible for the light to be On when all 503 switches are flipped down?

Solution: Answers, specifically the process of deriving solutions, may vary!

(a) Let A be one switch and let B be the other. 1 can correspond to being flipped up. 0 can correspond to being flipped down.

Furthermore, the light-bulb can have a boolean expression using the light switch variables and operators. An output of 1 can correspond to the light being On. An output of 0 can correspond to the light being Off.

(b) It would help to test all possible combinations of the switches being up and down and seeing how the light turns on and off accordingly.

We can record our observations in a table as follows where each row represents someone flipping exactly one switch at a time.

Switch A	Switch B	Light Output
Up	Down	On
Up	Up	Off
Down	Up	On
Down	Down	Off

We then represent switches as Boolean variables, A and B, where Up corresponds to 1 and Down corresponds to 0. Similarly the light output can have On correspond to 1 and Off correspond to 0. We can create the following truth table with a column for the unknown output.

A	B	?
1	0	1
1	1	0
0	1	1
0	0	0

If you reorder the rows, this truth table has the same values as the truth table for A XOR B so $(A \oplus B)$ (or any expression with the same evaluations as $(A \oplus B)$) can represent the light output.

(c) We'll use a similar strategy to part (b) but this time with a column for the third switch, C. All possible "observations"/combinations are as follows where each row represents someone flipping exactly one of the three switches at a time.

Switch A	Switch B	Switch C	Light Output
Up	Up	Up	On
Up	Up	Down	Off
Up	Down	Down	On
Up	Down	Up	Off
Down	Down	Up	On
Down	Up	Up	Off
Down	Up	Down	On
Down	Down	Down	Off

We find that in the case where all switches are Down, the light is Off. Therefore, it is impossible for the light to be On with all three switches Down.

Note, there is no "secret" combination of switch-flipping that leads to the light being On when all three switches are Down. The even/odd strategy described in part (d) explains why.

(d) We will not use a similar strategy to part (c) because a table of that size will not fit on the page. What we can do is examine patterns.

Referring back to the table of part (c), make note of the rows where the light is On. All of these rows have an odd number of switches flipped Up. Similarly, when the light is Off in the rows with an even number of switches flipped Up.

Since the initial conditions are essentially the same for the 503 switches, we can predict the state of the light-bulb based on the number of switches that are flipped up. If there is an odd number of switches that are Up, the light will be On. If there is an even number of switches that are Up, the light will be Off.

To answer the question, since 0 is an even number, there is an even number of switches flipped Up. Therefore, it is impossible for the light to be On with all 503 switches Down.