Grade 7/8 Math Circles
March 30th, 2022
Inequalities and Absolute Values

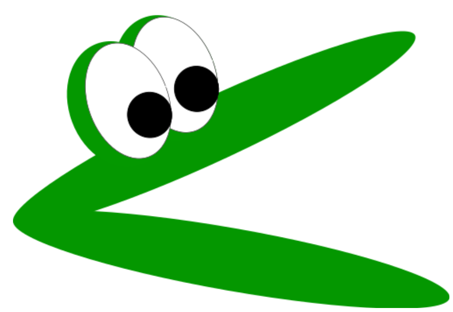
Introduction

You have likely seen equalities in your math classes before. They are usually represented with the symbol "=" and represent when two (or more) quantities are equal or the same. For example, we know that $2 + 2 = 4$. These are very helpful for us to understand many different mathematical and real-world concepts. However, not everything can be represented or understood using equalities. Sometimes, we want to use *inequalities*. Or, alternatively, we can sometimes use *absolute values*.

Defining Inequalities

Inequalities are when quantities are *greater than* or *less than* another quantity. We use the symbol "$<$" to represent "less than" and the symbol "$>$" to represent "greater than". A helpful way to remember the difference is that $<$ looks like a slanted L for "less"! Furthermore, when you're trying to decide which symbol to use in a specific equation, it can be helpful to visualize which direction the opening should face by using the following techniques:

- The smaller (or closed) side of the symbol faces the smaller number, and the bigger (or open) side faces the bigger number.
- Imagine the symbol is an alligator's mouth and it's deciding which quantity to eat. The alligator will always want the biggest meal, so his open mouth should face the bigger quantity to eat it!



Example 1
Since 5 is a bigger number than 3, we could write that $5 > 3$ or that $3 < 5$. Notice how we can swap the sides of the numbers, but this also means we need to swap the direction of the symbol.

Note that we don't always need exactly two numbers when we are using inequalities. Sometimes, we can use the symbols with just one number. For example, if someone wrote "Must be $>$ 18 years old", it would mean "Must be more than 18 years old". Furthermore, we can use the symbols in a *chain* to show how numbers compare to each other. For example, when we write "$6 < 9 < 11$", it means that $6 < 9$ *and* $9 < 11$. We write it in a chain, with all the inequality symbols facing the same way, so we can clearly see that 9 is between 6 and 11, and that the numbers get bigger from left to right. In general, when writing chains of inequalities, the symbols should always face the same way as it's easier to understand.

Exercise 1

Insert the correct symbols for the following equations/expressions.

(a) 15 ____ 10

(b) 3 ____ 8

(c) 20 ____ 6 ____ 3

(d) 23 ____ 43 ____ 63

(e) 52 ____ 26 ____ 14

(f) 50 ____ 80 ____ 100

We don't only use inequality symbols for representations, sometimes when we have an equation with an equality, we might want to solve for a variable to see when the inequality would be true. This requires us to know order of operations and how to solve regular equations!

Review of Order of Operations

If you have not seen order of operations before, or if you would like a review, please watch this YouTube video: https://youtu.be/ClYdw4d4OmA.

Review of Solving Equations

When we have one or more unknown variables in an equation, we may want to solve for what those variables have to equal to make the equation be true. Below are the steps you should follow to solve an equation.

1) Determine what you are trying to isolate/solve for.

2) Simplify the equation as much as possible by adding and subtracting *like terms*.

   Like terms are terms in a mathematical equation that have the exact same variables; only their *coefficients* are different. **Coefficients** are what we call the numbers that are attached to and in front of variables.

You can think of it like adding apples and oranges. If I have 1 apple plus 3 oranges plus 2 apples plus 1 orange, I actually have 3 apples and 4 oranges. A mathematical example with the variables x and y can be if we have the expression $x + 3y + 2x + y$, we can collect the like terms ($x + 2x$ and $3y + y$) to get $3x + 4y$.

3) Isolate the desired variable on one side of the equal sign and everything else on the other side by performing *opposite operations*. Note that it's usually best to do additions/subtractions first and then multiplications/divisions.

Original Operation		**Opposite Operation**	
Addition	$+$	$-$	Subtraction
Subtraction	$-$	$+$	Addition
Division	$\div$	$\times$	Multiplication
Multiplication	$\times$	$\div$	Division

When performing opposite operations, what you do to one side of the equation you **must** do to the other side of the equation. For example, if you want to subtract 2 from the left-hand-side (LHS) of the equal sign, you also need to subtract 2 from the right-hand-side (RHS) of the equal sign to balance it out.

Our end goal of isolating a variable—for example, the variable x—is to obtain an equation of the form "$x =$ ..." or "... $= x$". Notice that x is positive with a coefficient of 1, or in other words, there should not be a minus sign or other coefficients in front of x. Furthermore, there should be no other x's on the other side of the equal sign, because isolating x means having it by itself. Whatever x is equal to is what the value of x needs to be to make the equation true.

Example 2

Solve for x in the equation $3x - 1 + x + 6 = 2x - 11$.

We begin by collecting like terms on both sides. The RHS is already simplified, so we just need to simplify the LHS. We see that we have $3x + x$ and $-1 + 6$ on the LHS, and so after simplification we end up with $4x + 5 = 2x - 11$.

Now, we want to perform our opposite operations. First, let's move all the terms with x to the LHS. To move $2x$ to the LHS, we subtract $2x$ from both sides because $2x$ is like $+2x$. So, we get $4x + 5 - 2x = -11$.

Next, let's move everything without x from the LHS to the RHS. We can move the $+5$ over to the RHS by subtracting 5, and we get $4x - 2x = 11 - 5$.

Now, everything with x is on the LHS and everything else is on the RHS, so we can collect like terms again to make our calculations easier. After collecting like terms, we have $2x = 6$. Notice that x is multiplied by 2, and since we want x by itself, we should divide both sides by 2. This will give us that $x = 6 \div 2$, which we can further simplify to get our final answer that $x = 3$.

Exercise 2

Solve for x in the following equations.

(a) $7x = 3x + 5 + 2x - 3$

(b) $x + 6 + 2x = 3 + 4x$

Solving Inequalities

Now that we have reviewed order of operations and solving regular equations, we can look at solving inequalities.

When solving equations with equal signs the expression remains true as long as everything you do to one side, you also do to the other side. When solving inequalities, there are certain operations that have no effect on the direction of the inequality and there are some operations that cause the direction of the inequality to switch in order to keep the expression true. When we say the inequality switches, we mean that $>$ becomes $<$, or $<$ becomes $>$. Below is a table of operations categorized by their effect on the direction of the inequality.

Direction of the Inequality...

Stays the Same	Switches
Multiplying both sides by a positive number	Multiplying both sides by a negative number
Dividing both sides by a positive number	Dividing both sides by a negative number
Adding a number to both sides	Switching left and right sides
Subtracting a number from both sides	
Simplifying a side	

Example 3

What will happen to the direction of the inequality symbols when the following operations are applied to the inequalities?

(a) Subtracting 5 from both sides of $x + 5 > 7$.

The direction of the inequality will stay the same (the inequality will be $x > 2$).

(b) Dividing both sides $3x < 6$ by 3.

The direction of the inequality will stay the same (the inequality will be $x < 2$).

(c) Switching the left and right sides of $4x < 8$.

The direction of the inequality will switch (the inequality will be $8 > 4x$).

(d) Dividing both sides of $-3x > 12$ by -3.

The direction of the inequality will switch (the inequality will be $x < -4$).

Overall, to solve an inequality, you should follow the same steps as when solving an equation with an equality, with the additional step of paying attention to the direction of the inequality.

Note that similar to isolating a variable in equalities, we want to end up having the variable by itself on one side of the inequality. Having the variable on the LHS would look like "$x <$ ..." or "$x >$ ..." which makes it easier to read out like "x is less than..." or "x is greater than...", but having the variable on the RHS is also okay.

Lastly, when we have our final solution, we can realize that x is not a single number like with equalities, but rather can be *any* number that satisfies the condition! For example, we found in

Example 3 that the inequality (b) is satisfied as long as $x < 2$. So, it is satisfied by any number that is less than 2, like when $x = 1,\ x = 0,\ x = -0.1$, and so on.

Example 4

Solve for x in the following inequalities.

(a) $2x + 5 + x > 7 + x + 4$

We begin by collecting like terms on either side to get the equation $3x + 5 > 11 + x$. Then, we can move $+x$ to the LHS by subtracting x from both sides of the equation, giving us $3x + 5 - x > 11$. Next, we move over the $+5$ by subtracting 5 from both sides, giving us $3x - x > 11 - 5$. Now, we collect like terms to get $2x > 6$. Lastly, we divide both sides by 2 to get x by itself and we end up with the final answer that $x > 3$.

(b) $5 - 3x + x - 2x < 8 - 1 + 2$

We begin by collecting like terms on either side to get the equation $5 - 4x < 9$. Next, we can move the 5 over to the RHS by subtracting 5 from both sides, giving us $-4x < 4$ after simplification. So, now we can divide both sides by -4 to get x by itself, but remember that when we multiply by a negative number, we also need to swap the direction of the inequality sign. Therefore, we end up with the final answer that $x > -1$.

Exercise 3

Solve for x in the following inequalities.

(a) $4x - 8 < 20$

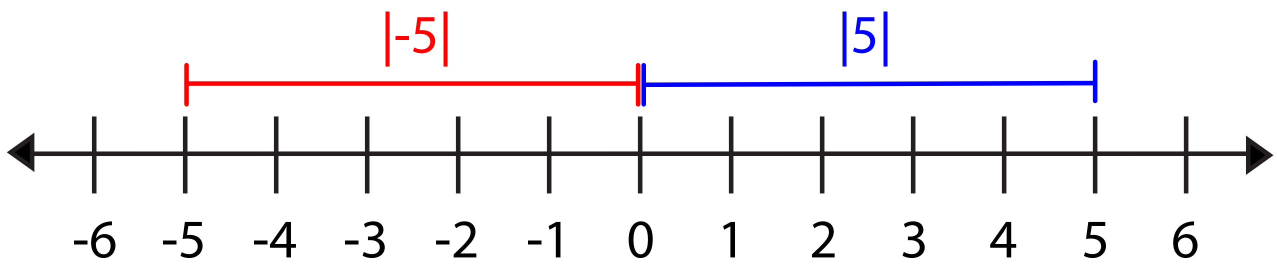
(b) $-2x + 3 > 10 + 2x + 9$

Defining Absolute Values

An **absolute value** is the distance between a number and zero, which can be visualized on a number line or a ruler. This means that all absolute values are positive numbers or zero, since distance can't be negative.

Absolute values are represented with the symbols $|\ |$ surrounding a number, similar to how brackets work. So, when we write $|5|$ or $|-5|$, it means we want to take the absolute value of 5 or -5.

Now, if we evaluate $|5|$ and $|-5|$, we see that $|5| = 5 = |-5|$. This is because 5 and -5 are both 5 units from 0, just in different directions, as seen on the number line below.



As you might have observed, absolute values will make the biggest difference for negative numbers. The absolute value of positive numbers will be the same number, whereas the absolute value of negative numbers will be the corresponding positive number. In other words, when we have that $x > 0$, $|x| = x$, and when $x < 0$, $|x| = -x$. For example, $|3| = 3$, and $|-3| = -(-3) = 3$. Furthermore, note that when $x = 0$, we get that $|x| = |0| = 0$.

Since the absolute value of a negative number is positive, absolute values can play an interesting role in subtraction as well as when multiplying or dividing by negative numbers.

When subtracting two numbers normally, the order in which the numbers are placed matters as it affects whether your answer is positive or negative. But, since negative numbers and their corresponding positive numbers are equal in terms of absolute value, the order of subtraction will not matter for an absolute value, like below.

$$|7 - 3| = |4| = 4$$

$$|3 - 7| = |-4| = 4$$

Then, when multiplying or dividing a positive number by a negative number normally, you would end up with a negative number. But, again since negative numbers and their corresponding positive number are equal in terms of absolute value, we get a positive number as our answer, like below.

$$|4 \times 5| = |20| = 20$$

$$|4 \times (-5)| = |-20| = 20$$

$$|4 \div 2| = |2| = 2$$

$$|4 \div (-2)| = |-2| = 2$$

The last thing that we should know about absolute values is that when absolute value are involved in an equation, they act similar to brackets, where you should calculate the absolute value of the inside of the | | before doing other operations. Furthermore, if there is a negative symbol *in front* of the absolute value symbols, we will always get a negative number, like below, since you calculate inside the absolute value symbols before applying the subtraction.

$$-|7-3| = -|4| = -4$$
$$-|3-7| = -|-4| = -4$$

Solving Absolute Values

When we have an unknown variable inside of an absolute value, we have to consider what it could be for the corresponding positive and negative numbers that give the same absolute value. By considering both options, absolute value equations will typically have two correct answers, like seen in the example below.

Example 5

Solve for x in the equation $|2x-1| = 5$.

Since the absolute value equals 5, it must be the case that $2x-1=5$ or that $2x-1=-5$. Therefore, we will solve both of these equations separately and the solution to each will be part of the final answer:

$$\begin{aligned} 2x-1 &= 5 \\ 2x &= 6 \\ x &= 3 \end{aligned} \qquad\qquad \begin{aligned} 2x-1 &= -5 \\ 2x &= -4 \\ x &= -2 \end{aligned}$$

So, the answer is that $x=3$ **or** $x=-2$. Note that it's very important to include the "or" in this statement, because x cannot be equal to both at the same time!

The same concept applies when there is an inequality involving absolute values. However, with inequalities, we need to consider if our answer should involve an "and" statement or an "or" statement. See how this works in the following example.

Example 6

Solve for x in the following inequalities.

(a) $|x| < 5$

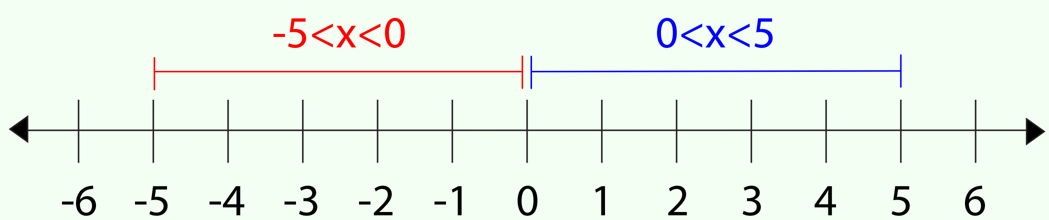
Recall that $|x| = 0$ when $x = 0$, $|x| = x$ when $x > 0$, and $|x| = -x$ when $x < 0$. So, in this problem, when $x = 0$, $|x| = 0$, and $0 < 5$ so the inequality is satisfied.

Then, when $x > 0$, our inequality will be equal to $x < 5$. Therefore, we know that $|x| < 5$ is satisfied when $x > 0$ *and* $x < 5$. We can write $x > 0$ and $x < 5$ as $0 < x < 5$ like we saw earlier since chains imply "and" statements.

Furthermore, when $x < 0$, our inequality will be equal to $-x < 5$. Multiplying both sides by -1, we see that $x > -5$. Notice that we switched the direction of the inequality because we multiplied by a negative number. Since $x < 0$ and $x > -5$, we know that $|x| < 5$ is also satisfied when $-5 < x < 0$.

Therefore, $|x| < 5$ is satisfied when $x = 0$, $0 < x < 5$, and $-5 < x < 0$. Combining these intervals, we can state that our final answer is $-5 < x < 5$.

For a visualization of this, we can look at the number line below. It's easy to see that the intervals join together when we also have $x = 0$, so that's why we simplify the intervals into $-5 < x < 5$.



(b) $|2x - 1| < 5$

When $2x - 1 = 0$, $|2x - 1| = 0$, and $0 < 5$ so it satisfies the inequality. Then, when $2x - 1 > 0$, our inequality will be equal to $2x - 1 < 5$. Lastly, when $2x - 1 < 0$, our inequality will be equal to $2x - 1 > -5$. Notice how we multiplied by -1 already.

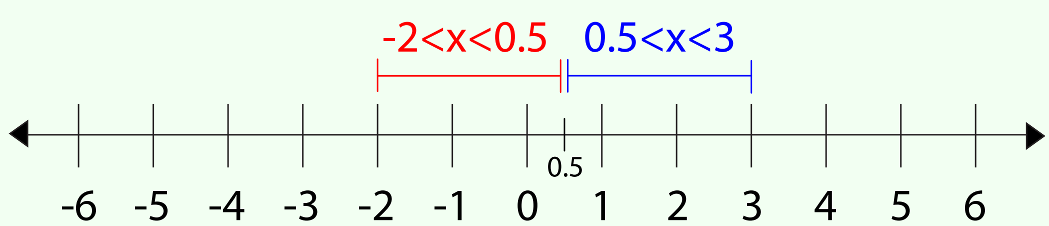
Multiplying by -1 right away can be easier or save time for many equations since the equation inside the absolute value will remain the same instead of having to figure out what it would be if it were negative. For example, in this case, we would have needed to calculate $-(2x-1) = -2x+1$. But instead, we can directly get the negative counterpart of $2x-1 < 5$ by switching the direction of the inequality and making the 5 negative to get $2x - 1 > -5$. You can check that this is just like multiplying the inequality $-2x+1 < 5$ by -1.

We must now solve for x in each of the conditions separately and then combine the answers afterwards:

$2x - 1 = 0$	$2x - 1 > 0$	$2x - 1 < 5$	$2x - 1 < 0$	$2x - 1 > -5$
$2x = 1$	$2x > 1$	$2x < 6$	$2x < 1$	$2x > -4$
$x = 0.5$	$x > 0.5$	$x < 3$	$x < 0.5$	$x > -2$

So, we got that when $2x - 1 = 0$, $x = 0.5$, and when $2x - 1 > 0$, $x > 0.5$ and $x < 3$. This means that $|2x - 1| < 5$ is satisfied when $x = 0.5$ or when $0.5 < x < 3$. Furthermore, we got that when $2x - 1 < 0$, $x < 0.5$ and $x < -2$. This means that $|2x - 1| < 5$ is also satisfied when $-2 < x < 0.5$. Combining all of these intervals, we get that our final answer is $-2 < x < 3$.

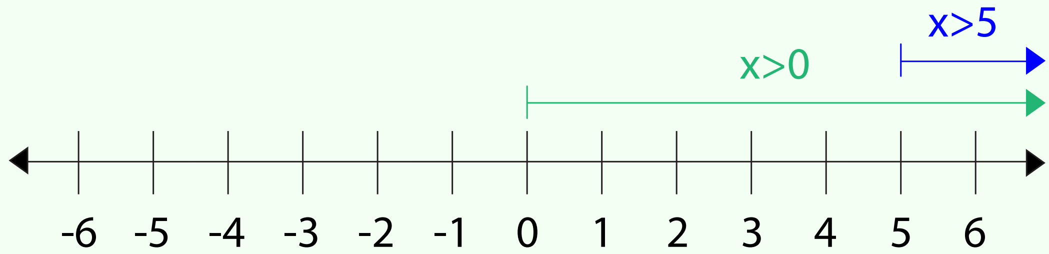
For a visualization of this, we can look at the number line below. It's easy to see that the intervals join together when we also have $x = 0.5$, so that's why we simplify the intervals into $-2 < x < 3$.



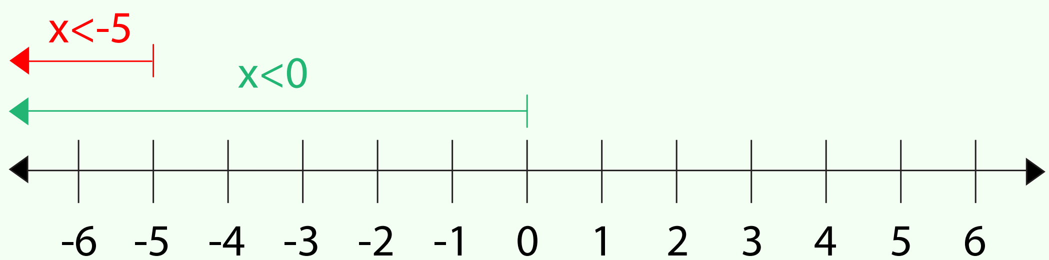
(c) $|x| > 5$

When $x = 0$, $|x| = 0$, and $0 \not> 5$ so it does not satisfy the inequality.

Then, when $x > 0$ our inequality will be equal to $x > 5$. Therefore, it must be the case that $x > 0$ and $x > 5$. If both $x > 0$ and $x > 5$, then since 5 is the bigger number, we can simplify this as just $x > 5$. We can also visually see why this is the case in the number line below.

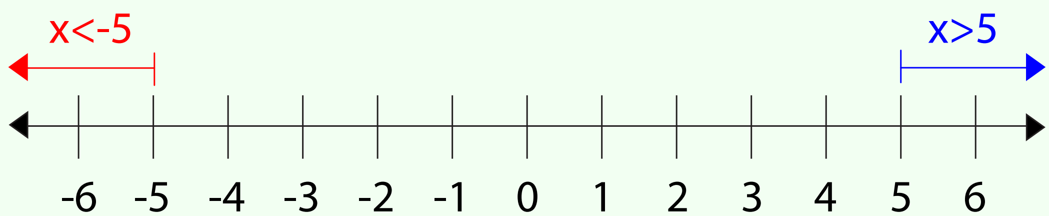


Furthermore, when $x < 0$, our inequality will be equal to $-x > 5$. Multiplying both sides by -1, we see that $x < -5$. Therefore, it must be the case that $x < 0$ and $x < -5$, which can be simplified as just $x < -5$ since -5 is the smaller number. We can also visually see why this is the case in the number line below.



Combining the two answers, we get that $x > 5$ **or** $x < -5$. Note that like in Example 5, we use "or" because while x will satisfy $|x| > 5$ in either condition, it's not possible for x to be in both conditions at the same time.

For a visualization of this answer, we can look at the number line below.



(d) $|2x - 1| > 5$

When $2x - 1 = 0$, $|2x - 1| = 0$, and $0 \not> 5$ so it does not satisfy the inequality.

Then, when $2x - 1 > 0$, our inequality will be equal to $2x - 1 > 5$. Therefore, it must be the case that $2x - 1 > 0$ and $2x - 1 > 5$. If both $2x - 1 > 0$ and $2x - 1 > 5$, then since 5 is the bigger number, we can simplify this as just $2x - 1 > 5$, similar to what we did in part (c).

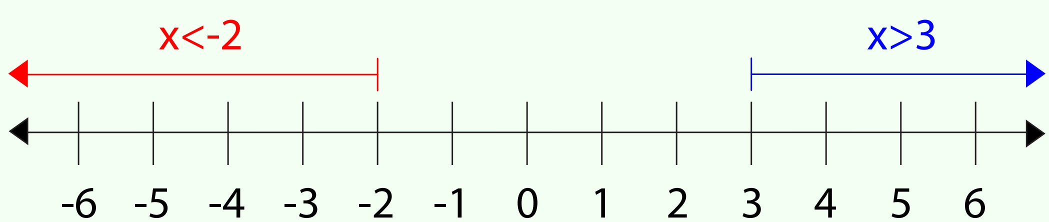
Furthermore, when $2x - 1 < 0$, our inequality will be equal to $2x - 1 < -5$. Notice how we multiplied by -1 already, like in part (b), by switching the inequality direction and making 5 negative. It must be the case that $2x - 1 < 0$ and $2x - 1 < -5$, which can be simplified as just $2x - 1 < -5$ since -5 is the smaller number, again similar to what we did in part (c).

Now, we must solve for x in each of these conditions separately and then combine the answers afterwards:

$$\begin{aligned} 2x - 1 &> 5 \\ 2x &> 6 \\ x &> 3 \end{aligned} \qquad\qquad \begin{aligned} 2x - 1 &< -5 \\ 2x &< -4 \\ x &< -2 \end{aligned}$$

So, we got that when $2x - 1 > 5$, $x > 3$, and when $2x - 1 < -5$, $x < -2$. Therefore, combining these two answers, we get that our final answer is $x > 3$ **or** $x < -2$. Note again that we use "or" because while x will satisfy $|2x - 1| > 5$ in either condition, it's not possible for x to be in both conditions at the same time.

For a visualization of this answer, we can look at the number line below.



Exercise 4

Solve for x in the following.

(a) $|2x + 5| = 15$

(b) $|x - 9| < 7$

(c) $|x + 5| > 7$

Conclusion

Inequalities can be very useful for representing scenarios, and solving for variables in inequalities allows you to find what range of answers are possible. Furthermore, absolute values are a very interesting concept and makes a big difference when evaluating expressions with negative numbers. Lastly, solving for variables in absolute value expressions can give you more than one answer, which you've likely never seen a mathematical equation have before, which is very cool. You likely won't see more complex scenarios involving inequalities and absolute values until late in high school, so knowing about them now gives you a great head start!