Grade 7/8 Math Circles
March 30th, 2022
Inequalities and Absolute Values Solutions

Exercise Solutions

Exercise 1

Insert the correct symbols for the following equations/expressions.

(a) 15 ____ 10

(b) 3 ____ 8

(c) 20 ____ 6 ____ 3

(d) 23 ____ 43 ____ 63

(e) 52 ____ 26 ____ 14

(f) 50 ____ 80 ____ 100

Exercise 1 Solution

(a) $15 > 10$

(b) $3 < 8$

(c) $20 > 6 > 3$

(d) $23 < 43 < 63$

(e) $52 > 26 > 14$

(f) $50 < 80 < 100$

Exercise 2

Solve for x in the following equations.

(a) $7x = 3x + 5 + 2x - 3$

(b) $x + 6 + 2x = 3 + 4x$

Exercise 2 Solution

(a) We collect like-terms to get $7x = 5x + 2$. Then, subtracting $5x$ from both sides, we get $2x = 2$. Finally, dividing both sides by 2, we get the final answer that $x = 1$.

(b) We collect like-terms to get $6 + 3x = 3 + 4x$. Then, subtracting $4x$ and 6 from both sides, we get $-1x = -3$. Now, dividing both sides by -1, we get the final answer that $x = 3$.

Alternatively, we could have subtracted $3x$ and 3 from both sides to give us $3 = x$. This way, we do not need to deal with negative numbers, although we will need to make sure to write that the final answer is $x = 3$ (not backwards) since it's easier to read.

Exercise 3

Solve for x in the following inequalities.

(a) $4x - 8 < 20$

(b) $-2x + 3 > 10 + 2x + 9$

Exercise 3 Solution

(a) We can add 8 to both sides to get $4x < 28$. Then, dividing by 4, we get that the final answer is $x < 7$.

(b) We collect like-terms to get $-2x + 3 > 2x + 19$. Then, subtracting 3 and $2x$ from both sides, we get $-4x > 16$. Now, dividing both sides by -4, we get that the final answer is $x < -4$. Notice how we switched the direction of the inequality because we divided by a negative number!

Alternatively, we could have added $2x$ to both sides and subtracted 19 from both sides to give us $-16 > 4x$, then dividing by 4 would give us $-4 > x$. This way, we do not divide by a negative number, but we need to remember to write that the final answer is $x < -4$ (not backwards) since it's easier to read.

Exercise 4

Solve for x in the following.

(a) $|2x + 5| = 15$

(b) $|x - 9| < 7$

(c) $|x + 5| > 7$

Exercise 4 Solution

(a) It must be the case that $2x + 5 = 15$ or $2x + 5 = -15$. Solving these separately, we get:

$$\begin{aligned} 2x + 5 &= 15 \\ 2x &= 10 \\ x &= 5 \end{aligned} \qquad\qquad \begin{aligned} 2x + 5 &= -15 \\ 2x &= -20 \\ x &= -10 \end{aligned}$$

So the final answer is that $x = 5$ **or** $x = -10$.

(b) When $x - 9 = 0$, $|x - 9| = 0$, and $0 < 7$ so the inequality is satisfied when $x = 9$. When $x - 9 > 0$, our inequality will be equal to $x - 9 < 7$. Adding 9 to both sides of both inequalities, we get $x > 9$ and $x < 16$. Therefore the inequality is satisfied when $9 < x < 16$. Lastly, when $x - 9 < 0$, our inequality will be equal to $x - 9 > -7$. Notice how we multiplied by -1 already. Adding 9 so both sides of these inequalities, we get $x < 9$ and $x > 2$. Therefore the inequality will be satisfied when $2 < x < 9$. Combining all of these intervals, our final answer is that $2 < x < 16$.

(c) When $x + 5 = 0$, $|x + 5| = 0$, and $0 \not> 7$ so the inequality is not satisfied. When $x + 5 > 0$, our inequality will be equal to $x + 5 > 7$. When both $x + 5 > 0$ and $x + 5 > 7$, we can simplify this to just $x + 5 > 7$ since 7 is the bigger number. Lastly, when $x + 5 < 0$, our inequality will be equal to $x + 5 < -7$. Notice how we multiplied by -1 already. When both $x + 5 < 0$ and $x + 5 < -7$, we can simplify this to just $x + 5 < -7$ since -7 is the smaller number. Solving for x in each of these conditions, we get:

$$\begin{aligned} x + 5 &> 7 \\ x &> 2 \end{aligned} \qquad\qquad \begin{aligned} x + 5 &< -7 \\ x &< -12 \end{aligned}$$

So, combining these two answers, the final answer is that $x > 2$ **or** $x < -12$.

Problem Set Solutions

1. Insert the correct inequality symbols to make each equation true.

   (a) 0 ____ 8

   (b) -5 ____ 4

   (c) $(5-7)$ ____ 0

   (d) $|-3|$ ____ $|-9|$

   (e) 0 ____ $|-2|$

   (f) -1 ____ $|-5|$

   > *Solution*:
   >
   > (a) $0 < 8$
   >
   > (b) $-5 < 4$
   >
   > (c) $(5-7) < 0$
   >
   > (d) $|-3| < |-9|$
   >
   > (e) $0 < |-2|$
   >
   > (f) $-1 < |-5|$

2. In a group of five friends:

   - Amy is taller than Carla
   - Eric is shorter than Ahmed but taller than Yin
   - Ahmed is shorter than Carla

   Use inequality symbols in a chain to list the friends from shortest to tallest.

   > *Solution*: Yin < Eric < Ahmed < Carla < Amy

3. Five children had dinner. Chris ate more than Max. Brandon ate less than Kayla. Kayla ate less than Max but more than Tanya. Use inequality symbols in a chain to list the children from who ate the most to who ate the least.

   > *Solution*: Chris > Max > Kayla > Tanya and Brandon
   > (There is not enough information to distinguish between Tanya and Brandon, we just know they both ate less than Kayla, so we put them together.)

4. Solve for x in the following inequalities.

(a) $4x > 12$

(b) $-2x < 8$

(c) $3x + 4 < 22$

(d) $2x + 9 < 6x + 1$

(e) $1 + 2x \times 2 + 2 < 5 + 2x + 2 - 2x$

(f) $2x + 4 + x - 3 < 4 - 2x + 3 + 5x$

Solution:

(a) Dividing both sides by 4, we get $x > 3$.

(b) Dividing both sides by -2, we get $x > -4$.

(c) Subtracting 4 from both sides, we get $3x < 18$. Then, dividing both sides by 3, we get $x < 6$.

(d) Subtracting 9 and $6x$ from both sides, we get $-4x < -8$. Then, dividing by -4 we get $x > 2$.

(e) Collecting like terms (including following BEDMAS), we get $3+4x < 7$. Subtracting 3 from both sides, we get $4x < 4$. Then, dividing both sides by 4, we get $x < 1$.

(f) Collecting like terms (including following BEDMAS), we get $3x + 1 < 3x + 7$. Subtracting $3x$ from both sides gives us $1 < 7$. This is always true for any value of x, therefore the solution is that x can be anything.

5. Solve for x in the following absolute value problems.

(a) $6 = |3x|$

(b) $|6x| > 18$

(c) $|x + 7| < 18$

(d) $9 > |6 - 3x|$

(e) $|3x + 2 \times 2| < 2 \times 5$

(f) $9 \div 3 < |3 + 2x \times 2 + 4|$

Solution:

(a) We need to solve both $3x = 6$ and $3x = -6$. Dividing both of these equations by 3, we get $x = 2$ and $x = -2$. Since x cannot be both of these values at the same time, our final answer should be $x = 2$ **or** $x = -2$.

(b) When $6x = 0$, $|6x| = 0$, and $0 \not> 18$ so the inequality is not satisfied. When $6x > 0$, the inequality equals $6x > 18$. Dividing both sides of both inequalities by 6, we get $x > 0$ and $x > 3$. Since both $x > 0$ and $x > 3$, we can simplify by just having $x > 3$ since it's the bigger number. Lastly, when $6x < 0$, the inequality equals $6x < -18$. Dividing both sides of both inequalities by 6, we get $x < 0$ and $x < -3$. Since both $x < 0$ and $x < -3$, we can simplify by just having $x < -3$ since it's the smaller number. Combining all these intervals where the inequality is satisfied, we get the final answer $x > 3$ **or** $x < -3$.

(c) When $x + 7 = 0$, we can solve that $x = -7$, and we know $|x + 7| = 0$ and $0 < 18$, so the inequality is satisfied when $x = -7$. When $x + 7 > 0$, the inequality equals $x + 7 < 18$. Subtracting 7 from both sides of both inequalities, we get $x > -7$ and $x < 11$. Therefore we have that the inequality is satisfied when $-7 < x < 11$. Lastly, when $x + 7 < 0$, the inequality equals $x + 7 > -18$. Subtracting 7 from both sides of both inequalities, we get $x < -7$ and $x > -25$. Therefore we have that the inequality is also satisfied when $-25 < x < -7$. Combining all these intervals where the inequality is satisfied, we get the final answer $-25 < x < 11$.

(d) When $6 - 3x = 0$, we can solve that $x = 2$, and we know $|6 - 3x| = 0$ and $0 < 9$, so the inequality is satisfied when $x = 2$. When $6 - 3x > 0$, the inequality equals $6 - 3x < 9$. Subtracting 6 from both sides of both inequalities, we get $-3x > -6$ and $-3x < 3$. Now, dividing both sides of both inequalities by -3, we get $x < 2$ and $x > -1$. Therefore we have that the inequality is satisfied when $-1 < x < 2$. Lastly, when $6 - 3x < 0$, the inequality equals $6 - 3x > -9$. Subtracting 6 from both sides of both inequalities, we get $-3x < -6$ and $-3x > -15$. Now, dividing both sides of both inequalities by -3, we get $x > 2$ and $x < 5$. Therefore we have that the inequality is satisfied when $2 < x < 5$. Combining all these intervals where the inequality is satisfied, we get the final answer $-1 < x < 5$.

(e) We start by simplifying the inequality to be $|3x + 4| < 10$. When $3x + 4 = 0$, we can solve that $x = -\frac{4}{3}$, and we know $|3x + 4| = 0$ and $0 < 5$, so the inequality is satisfied when $x = -\frac{4}{3}$. When $3x + 4 > 0$, the inequality equals $3x + 4 < 10$. Subtracting 4 from both sides of both inequalities, we get $3x > -4$ and $3x < 6$. Now, dividing both sides of both inequalities by 3, we get $x > -\frac{4}{3}$ and $x < 2$. Therefore we have that the inequality is satisfied when $-\frac{4}{3} < x < 2$. Lastly, when $3x + 4 < 0$, the inequality equals $3x + 4 > -10$. Subtracting 4 from both sides of both inequalities, we get $3x < -4$ and $3x > -14$. Now, dividing both sides of both inequalities by 3, we get $x < -\frac{4}{3}$ and $x > -\frac{14}{3}$. Therefore we have that the inequality is also satisfied when $-\frac{14}{3} < x < -\frac{4}{3}$. Combining all these intervals where the inequality is satisfied, we get the final answer $-\frac{14}{3} < x < 2$.

(f) We start by simplifying the inequality to be $3 < |7 + 4x|$. When $7 + 4x = 0$, $|7 + 4x| = 0$ and $0 \not> 3$ so the inequality is not satisfied. When $7 + 4x > 0$, the inequality equals $7 + 4x > 3$. Subtracting 7 from both sides of both inequalities, we get $4x > -7$ and $4x > -4$. Now, dividing both sides of both inequalities by 4, we get $x > -\frac{7}{4}$ and $x > -1$. We can simplify by having just $x > -1$ since -1 is bigger than $-\frac{7}{4} = -1.75$. Lastly, when $7 + 4x < 0$, the inequality is equal to $7 + 4x < -3$. Subtracting 7 from both sides of both inequalities, we get $4x < -7$ and $4x < -10$. Now, dividing both sides of both inequalities by 4, we get $x < -\frac{7}{4}$ and $x < -\frac{10}{4}$. We can simplify by having just $x < -\frac{5}{2}$ since $-\frac{10}{4} = -\frac{5}{2} = -2.5$ is smaller than $-\frac{7}{4} = -1.75$. Combining all these intervals where the inequality is satisfied, we get the final answer $x > -1$ **or** $x < -\frac{5}{2}$.

6. Solve for x in the following.

(a) $3 < x \div 2$

(b) $x \div 2 + 9 < 12$

(c) $4 < (x + 6) \div 3$

(d) $|(3x + 6) \div 2| < 6$

(e) $-2 < 6x - 2 < 4$

(f) $2 > x \div 2 + 3$ or $x \div 2 + 3 > 5$

Solution:

(a) Multiplying both sides by 2, we get $6 < x$. Swapping the sides, we get the final answer $x > 6$.

(b) Subtracting 9 from both sides, we get $x \div 2 < 3$. Multiplying both sides by 2, we get $x < 6$.

(c) Multiplying both sides by 3, we get $12 < x + 6$. Subtracting 6 from both sides, we get $6 < x$, and then we can swap sides to get the final answer $x > 6$.

(d) When $(3x+6) \div 2 = 0$, we can solve that $x = -2$, and we know that $|(3x+6) \div 2| = 0$ and $0 < 6$, so the inequality is satisfied when $x = -2$. When $(3x + 6) \div 2 > 0$, the inequality is equal to $(3x+6) \div 2 < 6$. Multiplying both sides of both inequalities by 2, we get $3x + 6 > 0$ and $3x + 6 < 12$. Then, subtracting 6 from both sides of both inequalities, we get $3x > -6$ and $3x < 6$. Now, dividing both sides by 3, we get $x > -2$ and $x < 2$. Therefore the inequality is satisfied when $-2 < x < 2$. Lastly, when $(3x+6) \div 2 < 0$, the inequality equals $(3x+6) \div 2 > -6$. Multiplying both sides of both inequalities by 2, we get $3x + 6 < 0$ and $3x + 6 > -12$. Then, subtracting 6 from both sides of both inequalities, we get $3x < -6$ and $3x > -18$. Now, dividing both sides by 3, we get $x < -2$ and $x > -6$. Therefore the inequality is satisfied when $-6 < x < -2$. Combining all these intervals where the inequality is satisfied, we get the final answer $-6 < x < 2$.

(e) We can solve the inequalities separately then combine the answers. First, we can solve $6x - 2 > -2$. We add 2 to both sides to get $6x > 0$. Then, dividing both sides by 6, we get $x > 0$. Next, we can solve $6x - 2 < 4$. We add 2 to both sides to get $6x < 6$. Then, dividing both sides by 6, we get $x < 1$. So we have $x > 0$ and $x < 1$, which we can simplify into the final answer $0 < x < 1$.

(f) We can solve the inequalities separately and then combine the answers. First, we can solve $x \div 2 + 3 < 2$. We subtract 3 from both sides to get $x \div 2 < -1$. Then, we multiply both sides by 2 to get $x < -2$. Next, we can solve $x \div 2 + 3 > 5$. We subtract 3 from both sides to get $x \div 2 > 2$. Then, multiplying both sides by 2, we get $x > 4$. Combining these answers, we see that x can't be in both intervals at once, so our final answer is $x < -2$ **or** $x > 4$.

7. Based on your knowledge of the equal sign and inequality symbols, what do you think is the meaning of the symbols $\leq$ and $\geq$?

Solution: The meaning of $\leq$ is "less than or equal to" and the meaning of $\geq$ is "greater than or equal to". Essentially, these are combinations of equality and inequality. For example, if we said that $x \geq 6$, then $x = 6$ or $x > 6$. Furthermore, if we said that $x \leq 0$, then $x = 0$ or $x < 0$.

8. Simplify the following lists of conditions for x into an "and" or an "or" statement of two conditions.

   (a) $x > 3,\ x > 4,\ x < -2,\ x < 0$

(b) $x > -8,\ x > -3,\ x < 9,\ x < 1$

(c) $|-2x-2| > 4,\ 3x-4 > 5,\ |5x| > 15$

Solution:

(a) $x > 4$ or $x < -2$

(b) $x > -3$ and $x < 1$ (alternatively could use a chain: $-3 < x < 1$)

(c) $x < -3$ or $x > 3$

9. Try to explain the reasoning behind the fact that $|x| = \sqrt{x^2}$.

Solution: When we input a positive number or zero into $|x|$, we know that we will get the same positive number. Furthermore, when we input a positive number into $\sqrt{x^2}$, we will get the same positive number because we are simply squaring and then square rooting the positive number, which cancel out.

When we input a negative number into $|x|$, we know that we will get the positive equivalent. Furthermore, when we put a negative number into $\sqrt{x^2}$, first it gets squared, which makes it positive (since multiplying two negatives gives a positive). Then, the square root of this positive number is taken, which will give us the original number, but positive. So we end up with the positive equivalent of the negative number.

These are all the possible scenarios, and we saw that both functions act the same in all the scenarios, therefore they must be the same.

10. **Challenge:** Solve for x in the inequality $|x-9| < |x+5|$.

Solution: Let's start by looking at when $x - 9 = 0$. Solving for x, we get that $x = 9$. Substituting this value for x into our inequality, we get $|9-9| < |9+5|$, which simplifies to $0 < 14$. It's true that $0 < 14$, therefore our inequality is satisfied when $x = 9$.

Next, let's look at when $x - 9 > 0$. Solving for x, we get that $x > 9$, and since $x - 9 > 0$, we can simplify $|x-9| = x-9$ in the original inequality. Therefore, now we just need to solve $x - 9 < |x+5|$, given that $x > 9$.

- When $x + 5 = 0$, we can see that $x = -5$. But, we require that $x > 9$, so this case is not possible.

- When $x + 5 > 0$, we can see that $x > -5$. So, we have that $x > 9$ and $x > -5$, which we can simplify as just $x > 9$ for this case. Furthermore, since $|x+5| > 0$, we can simplify $|x+5| = x+5$ in our inequality. Therefore, we can solve $x-9 < x+5$. Subtracting x from both sides, we get $-9 < 5$, which is true, so our inequality is satisfied when $x > 9$.
- When $x + 5 < 0$, we can see that $x < -5$. But, we require that $x > 9$, so this case is not possible.

To summarize, we have that when $x - 9 > 0$, the inequality is only satisfied when $x > 9$.

Now, let's look at when $x - 9 < 0$. Solving for x, we get that $x < 9$, and since $x - 9 < 0$, we can simplify $|x-9| = -(x-9) = -x+9 = 9-x$ in the original inequality. Therefore, we just need to solve $9 - x < |x+5|$, given that $x < 9$.

- When $x + 5 = 0$, we can see that $x = -5$. Substituting this value for x into our inequality, we get $9 - (-5) < |(-5) + 5|$ which simplifies to $14 < 0$. It is not true that $14 < 0$, so our inequality is not satisfied in this case.
- When $x + 5 > 0$, we can see that $x > -5$. Furthermore, since $|x + 5| > 0$, we can simplify $|x + 5| = x + 5$ in our inequality. Therefore, we can solve $9 - x < x + 5$. Adding x to both sides and subtracting 5 from both sides, we get $4 < 2x$. Then, dividing by 2 on both sides gives us $2 < x$, which we swap sides to read as $x > 2$. So, we have that $x < 9$, $x > -5$, and $x > 2$ in this case, which simplifies to our inequality being satisfied when $2 < x < 9$.
- When $x + 5 < 0$, we can see that $x < -5$. Furthermore, since $|x + 5| < 0$, we can simplify $|x + 5| = -(x + 5) = -x - 5$ in our inequality. Therefore, we can solve $9-x < -x-5$. Adding x to both sides gives us $9 < -5$, which is not true, therefore our inequality is not satisfied in this case.

To summarize, we have that when $x-9 < 0$, the inequality is only satisfied when $2 < x < 9$.

We have now finished looking at all possible cases, and we can see that the inequality is satisfied when $x = 9$, $x > 9$, and $2 < x < 9$. Combining these intervals, we get that our final answer is $x > 2$.