Grade 9/10 Math Circles

February 23, 2022

Linear Diophantine Equations Part 2 - Problem Set

1. This problem will step you through determining all non-negative solutions to the linear Diophantine equation $12x + 57y = 423$.
   (a) Use the Euclidean Algorithm to calculate $\gcd(12, 57)$.
   (b) Using part (a), determine a solution to $12x + 57y = 3$.
   (c) Using part (b), determine a solution to $12x + 57y = 423$.
   (d) Using part (c), determine *all* solutions to $12x + 57y = 423$.
   (e) Using your answer in part (d), determine all solutions to $12x + 57y = 423$ with $x \geq 0$ and $y \geq 0$. That is, determine all non-negative solutions to the linear Diophantine equation $12x + 57y = 423$.
2. Explain why there is no solution to the linear Diophantine equation from Exercise 2,

$$4182x + 3689y = 102$$

   with $x \geq 0$ and $y \geq 0$.
3. Determine all possible ways that 1000 can be expressed as the sum of two **positive** integers, one which is divisible by 11 and the other by 17.
4. At a museum, an adult ticket costs \$34 and a student ticket costs \$28. A group visiting the museum spends exactly \$844 on tickets. Determine all possible combinations for the number of adult and student tickets they could have purchased.
5. Find the smallest positive integer x so that $157x$ leaves remainder 10 when divided by 24.
6. Determine the number of ways you can make exactly \$200 using exactly 1000 coins if each coin is a quarter, a dime, or a nickel.
7. Let a, b, and c be positive integers and consider the linear Diophantine equation $ax + by = c$. Show that the number of non-negative integer solutions to this equation cannot exceed $\frac{c}{a}$ or $\frac{c}{b}$.