Grade 9/10 Math Circles
March 30, 2022
Knot theory - Problem Set

This worksheet consists of many problems, which are divided by topic. Do the ones that seem the most interesting to you!

Alternating knots

An **alternating** knot is a knot with a diagram in which the crossings alternate between over and under as one travels around the knot in a fixed direction. For example, the diagram for the trefoil knot with three crossings is alternating.

1. Which diagrams in the knot table are alternating?
2. Find a diagram for the figure eight knot which is not alternating.
3. It turns out that "most" knots are not alternating (although many knots with a small number of crossings are). Show that you can change the crossings of any knot diagram to produce an alternating diagram (for a different knot).

   Careful: this is not as easy as one might think. Why does your argument always work?

Links

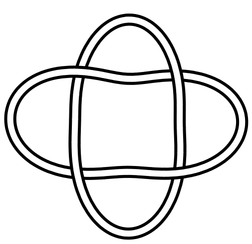


Figure 1: The Whitehead link.

The next few questions deal with links, which are a generalization of knots (some are illustrated below). A **link** is a collection of knots, which may be tangled together. The number of knots in a link is called the number of **components** of the link. The n-component *unlink* is the simplest possible link: it consists of n unknots, which are not tangled together.

For example, the illustration on the left describes one of the simplest 2-component links which is not the unlink, called the *Whitehead link*.

1. A (nontrivial) n-component link is called **Brunnian** if the removal of *any* component produces an $(n-1)$ component unlink. For example, the Whitehead link is a 2-component Brunnian link. Can you find a 3-component Brunnian link?

2. For each natural number n, find an n-component Brunnian link.

3. The following knot is called a **Pretzel link**, and is denoted $P(-3, 4, 2)$. The top and bottom are connected by three tangles which twist clockwise or counterclockwise (depending on the sign of each entry)

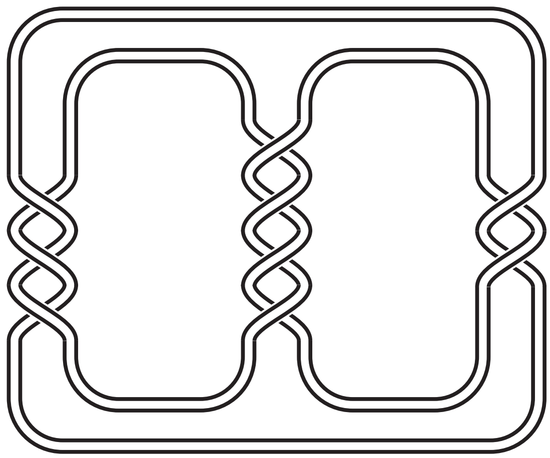


Figure 2: The link $P(-3, 4, 2)$.

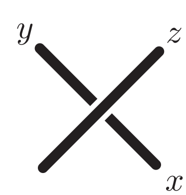
For any integers a, b, and c, we can produce an analogous link with a, b, and c twists. For what values of a, b, and c is $P(a, b, c)$ actually a knot?

Tricolorability

1. Show that the trefoil knot and the figure eight knot are distinct knots.

   Hint: Think about tricolorability.

2. In this question. we will consider a generalization of tricolorabiliy called an n-coloring. In particular, this will let us show that the figure eight knot cannot be unknotted.
   (a) Show that you can label the strands of the figure-eight knot with the integers $1, 2, 3, 4$ (using at least two distinct integers) so that at each crossing, the number $x + y - 2z$ is divisible by 5 (where x, y, and z are as in the illustration).

   (b) If a knot can be labelled in such a way, we say that it admits a **4-coloring**. You may take for granted that this property is a knot invariant (although you can also try to prove it using Reidemeister moves!). Show that the unknot does *not* admit a Fox 4-coloring, and conclude that the figure eight knot and the unknot are distinct knots.