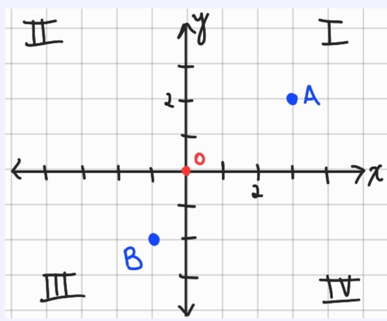
Grade 9/10 Math Circles
March 9, 2022
Analytic Geometry - Solutions

Exercise Solutions

Exercise 1

(a) What are the coordinates of B on the diagram below?

(b) What are the coordinates of the point that results from a reflection of B across the x-axis and then the y-axis?



Exercise 1 Solution

(a) B is -1 along the x-axis and -2 along the y-axis, so B is at $(-1, -2)$.

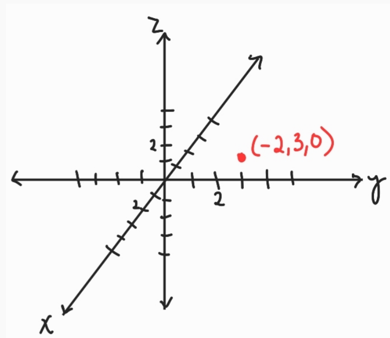
(b) The reflection across the x-axis takes the point to $(-1, 2)$, and then the further reflection across the y-axis takes the point to $(1, 2)$.

Exercise 2

What are the three dimensional coordinates of the point that corresponds with the location of $(-2, 3)$ in two dimensional space? Draw this point.

Exercise 2 Solution

Bringing the point $(-2, 3)$ into three dimensional space would mean setting its z coordinate to zero (so that it still sits on the xy-plane), so the point is $(-2, 3, 0)$. This is drawn below.



Exercise 3

The points $(1, 2)$ and $(5, p)$ are a distance of 5 away from each other. Determine the possible values of p.

Exercise 3 Solution

Using the distance formula, we find

$$\begin{aligned} 5 &= \sqrt{(1-5)^2 + (2-p)^2} \\ 25 &= (-4)^2 + (2-p)(2-p) \\ 25 &= 16 + 4 - 4p + p^2 \\ 0 &= p^2 - 4p - 5 \\ 0 &= (p-5)(p+1) \end{aligned}$$

Giving us $p = 5$ and $p = -1$.

Exercise 4

Determine the centre of a circle which passes through points $P(0,4)$, $Q(2,0)$, and $R(9,1)$.
Hint: The radius of a circle is constant.

Exercise 4 Solution

Let's call the centre of the circle the point $C(x_c, y_c)$. Since the radius of a circle is constant, each of P, Q, and R are the same distance from C. That is, $d_{PC} = d_{QC} = d_{RC}$.
We start by using the first part of the equality

$$\begin{aligned}
d_{PC} &= d_{QC} \\
\sqrt{(0-x_c)^2+(4-y_c)^2} &= \sqrt{(2-x_c)^2+(0-y_c)^2} \\
(-x_c)^2+(4-y_c)^2 &= (2-x_c)^2+(-y_c)^2 \\
x_c^2+16-8y_c+y_c^2 &= 4-4x_c+x_c^2+y_c^2 \\
4x_c-8y_c &= -12 \qquad (1)
\end{aligned}$$

And next we use the second part of the equality

$$\begin{aligned}
d_{QC} &= d_{RC} \\
\sqrt{(2-x_c)^2+(0-y_c)^2} &= \sqrt{(9-x_c)^2+(1-y_c)^2} \\
(2-x_c)^2+(-y_c)^2 &= (9-x_c)^2+(1-y_c)^2 \\
4-4x_c+x_c^2+y_c^2 &= 81-18x_c+x_c^2+1-2y_c+y_c^2 \\
14x_c+2y_c &= 78 \qquad (2)
\end{aligned}$$

And solving the system of two equations and two unknowns (1) and (2) then yields $x_c = 5$ and $y_c = 4$, so the centre is thus at $C(5,4)$.

Exercise 5

Determine the values of p for which the point $(1,1,p)$ is the smallest possible integer distance away from the origin.

Exercise 5 Solution

Making use of the three-dimensional distance formula with the points $(0, 0, 0)$ and $(1, 1, p)$ yields

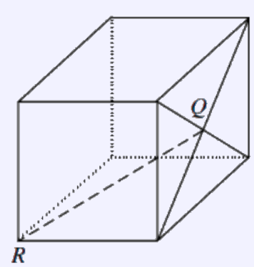
$$\begin{aligned} d &= \sqrt{(1-0)^2 + (1-0)^2 + (p-0)^2} \\ d &= \sqrt{1+1+p^2} \\ d &= \sqrt{2+p^2} \end{aligned}$$

Now, for d to be an integer, this means that $2 + p^2$ must be a perfect square. Noting that p^2 is always non-negative, we know that $2 + p^2$ must be at least 2. Then, the smallest perfect square greater than 2 is 4. Thus, we have

$$\begin{aligned} 4 &= 2 + p^2 \\ 2 &= p^2 \\ \pm\sqrt{2} &= p \end{aligned}$$

Exercise 6

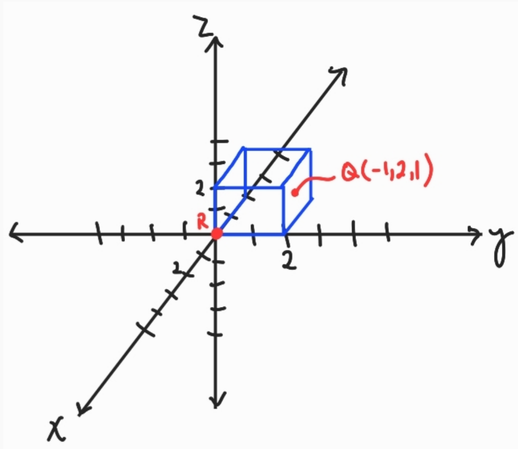
Q is the point of intersection of the diagonals of one face of a cube of side length 2. What is the length of QR?



1998 Pascal Contest, Q21. Hint: Place R at the origin.

Exercise 6 Solution

We place the cube with R at the origin as in the diagram below.



Now, Q is on the right face of the cube, so its y-coordinate must be 2 (the length of the cube). Being in the centre of its face, Q is halfway along the sides of the face of the cube it sits on (1 unit), so the x-coordinate is thus -1 and the z coordinate is thus 1 from our setup.
This problem thus boils down to finding the distance between $(0, 0, 0)$ and $(-1, 2, 1)$, which we find using the distance formula as

$$d_{QR} = \sqrt{(-1-0)^2 + (2-0)^2 + (1-0)^2)}$$
$$d_{QR} = \sqrt{6}$$

Exercise 7

Find the coordinates of the point that is $\frac{1}{4}$ of the distance from $A(-1, -1)$ to $B(3, 7)$.

Exercise 7 Solution

The point that is $\frac{1}{4}$ of the distance from A to B is the midpoint of the midpoint of AB.
Thus, we first find

$$M_{AB} = \left(\frac{-1+3}{2}, \frac{-1+7}{2}\right)$$
$$M_{AB} = (1, 3)$$

And we next find

$$M_{AM} = \left(\frac{-1+1}{2}, \frac{-1+3}{2}\right)$$
$$M_{AM} = (0, 1)$$

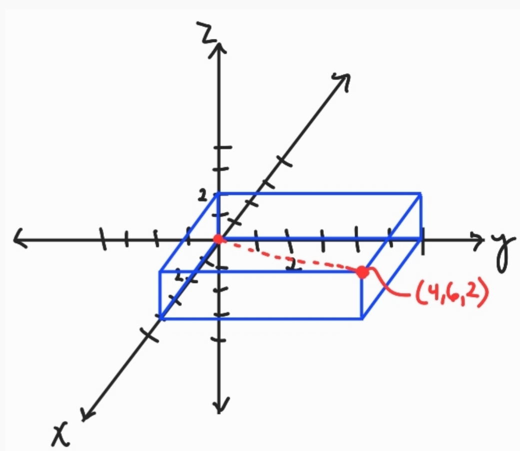
So, the point $(0, 1)$ is $\frac{1}{4}$ the distance from A to B.

Exercise 8

Find the coordinates of the centre of a rectangular prism of height 2 along the z-axis, width 4 along the x-axis, and length 6 along the y-axis, residing in the positive regions of the x, y, and z axes, and whose corner is at the origin.

Exercise 8 Solution

The setup of the question results in the diagram below.



Now, the centre of the prism resides halfway along the internal diagonal of the prism, which runs from $(0, 0, 0)$ to $(4, 6, 2)$. We thus find the midpoint of the diagonal, and thus by extension the centre of the prism, as

$$\begin{aligned} M &= \left(\frac{0+4}{2}, \frac{0+6}{2}, \frac{0+2}{2}\right) \\ M &= (2, 3, 1) \end{aligned}$$

Exercise 9

$A(-3, 2)$, $B(7, 2)$, and $C(-1, p)$ form a right-angled triangle with the right angle at point C. Determine all possible values of p, without using distance formulae or Pythagorean Theorem.

Exercise 9 Solution

Since the right angle is at point C, this means that the line (segments) AC and BC are perpendicular. Thus, their slopes must be negative reciprocals.

We focus first on finding the slope of the line connecting A to C:

$$\begin{aligned} m_{AC} &= \frac{p-2}{-1-(-3)} \\ m_{AC} &= \frac{p-2}{2} \end{aligned}$$

And now the slope of the line connecting B to C:

$$\begin{aligned} m_{BC} &= \frac{p-2}{-1-7} \\ m_{BC} &= \frac{p-2}{-8} \end{aligned}$$

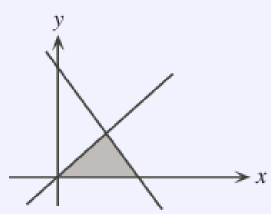
And, using the fact that the slopes are negative reciprocals we have

$$\begin{aligned} m_{AC}m_{BC} &= -1 \\ \frac{1}{2}(p-2)\frac{-1}{8}(p-2) &= -1 \\ p^2 - 4p + 4 &= 16 \\ p^2 - 4p - 12 &= 0 \\ (p-6)(p+2) &= 0 \end{aligned}$$

Thus giving us that $p = 6$ or $p = -2$.

Exercise 10

In the diagram, the shaded region is bounded by the x-axis and the lines $y = x$, and $y = -2x+3$. What is the area of the shaded region?



2008 Fermat Contest, Question 10

Exercise 10 Solution

We first note that the x-intercepts of the lines are at 0 and $\frac{3}{2}$ by substituting $y = 0$ and solving for x in both linear equations. Thus, the base of the triangle is $\frac{3}{2}$.
Next, we find the point of intersection of the lines

$$\begin{aligned} x &= -2x + 3 \\ 3x &= 3 \\ x &= 1 \end{aligned}$$

and upon substitution into $y = x$ (since this one is easier to work with), we find $y = 1$. Thus, the point of intersection is $(1, 1)$.
The y-coordinate of this point represents the height of the triangle, so the height is 1.
Then, the area of the triangle is $\frac{1}{2}(\frac{3}{2})(1) = \frac{3}{4}$.

Exercise 11

If $A(2, 5)$, $B(-3, -6)$, and $C(p, 0)$ form a triangle of area 10, determine the possible values of p.
Hint: Note that $|x| = a$ leads to cases $x = a$ or $x = -a$.

Exercise 11 Solution

Substitution into the area formula gives

$$\begin{aligned} 10 &= \frac{1}{2}|(p)(-6) - (p)(5) - (2)(-6) + (-3)(5) - (-3)(0) + (2)(0)| \\ 20 &= | - 11p - 3| \end{aligned}$$

This leads to the two possible cases of $20 = -11p - 3$ or $-20 = -11p - 3$.
Solving the first case leads to $p = -\frac{23}{11}$ while solving the second leads to $p = \frac{17}{11}$.

Problem Set Solutions

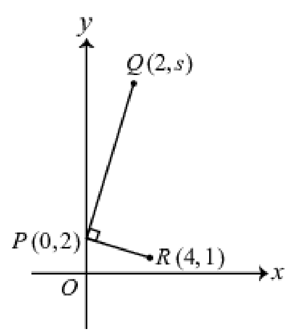
1. *(2018 Cayley, Question 8)* For what value of k is the line through the points $(3, 2k + 1)$ and $(8, 4k - 5)$ parallel to the x-axis?

   Solution: For a line to be parallel to the x-axis, it must be horizontal, and thus has slope zero.

We compute the slope between the provided points as

$$0 = \frac{(4k-5)-(2k+1)}{8-3}$$
$$0 = \frac{2k-6}{5}$$
$$0 = 2k-6$$
$$6 = 2k$$
$$3 = k$$

2. *(2019 Cayley, Question 15)* In the diagram below, the line segments PQ and PR are perpendicular. What is the value of s?



Solution: For the line segments to be perpendicular, their slopes must be negative reciprocals.
We focus first on finding the slope of the line connecting P to R:

$$m_{PR} = \frac{2-1}{0-4}$$
$$m_{PR} = -\frac{1}{4}$$

So the slope of the line connecting P to Q must be 4. We thus have

$$\begin{aligned} 4 &= \frac{s-2}{2-0} \\ 4 &= \frac{s-2}{2} \\ 8 &= s-2 \\ 10 &= s \end{aligned}$$

3. Determine the values of a and b so that the lines $y = 3ax - b$ and $y = -2x + a$ intersect at $(1, 1)$.

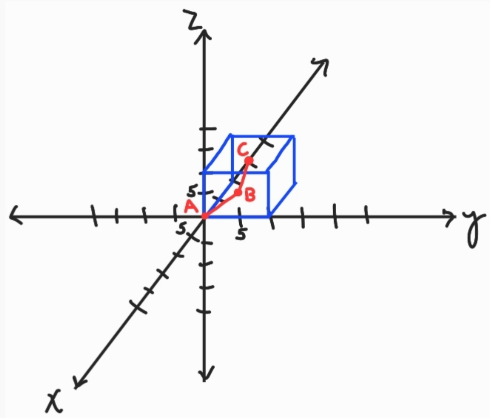
Solution: Since the lines intersect at $(1, 1)$, they each pass through $(1, 1)$. Substituting $x = 1$ and $y = 1$ into both equations yields the two equations with two unknowns

$$\begin{aligned} 1 &= 3a - b \\ 1 &= -2 + a \end{aligned}$$

The second equation immediately provides $a = 3$, and then substitution into the first yields $b = 8$.

4. We have a cheese cube of side length 10. An ant begins in one of the bottom corners, and walks diagonally across one of the side faces to its centre. The ant then burrows into the cheese on a diagonal trajectory until it pops out at the centre of the top face. How far was the ant's total journey on this path?

Solution: We set up the cube and draw the path as depicted below.



The ant's travel is broken into two segments, the walk from $A(0,0,0)$ to $B(0,5,5)$ and the walk from $B(0,5,5)$ to $C(-5,5,10)$. Using the distance formula in three dimensions, we have

$$\begin{aligned} d_{AB} + d_{BC} &= \sqrt{(0-0)^2 + (5-0)^2 + (5-0)^2} + \sqrt{(-5-0)^2 + (5-5)^2 + (10-5)^2} \\ &= \sqrt{50} + \sqrt{50} \\ &= 10\sqrt{2} \end{aligned}$$

5. Given the triangle formed by the points $A(a,0)$, $B(0,b)$, and $C(0,0)$, show that the triangle formed by the midpoints of the sides of $\triangle ABC$ is one quarter the area of $\triangle ABC$.

Solution: We can use the area formula to find the area of $\triangle ABC$

$$\begin{aligned} A_{\triangle ABC} &= \frac{1}{2}|(0)b - (0)(0) - ab + (0)(0) - (0)(0) + a(0)| \\ &= \frac{1}{2}ab \end{aligned}$$

Next, we find the midpoints of the sides

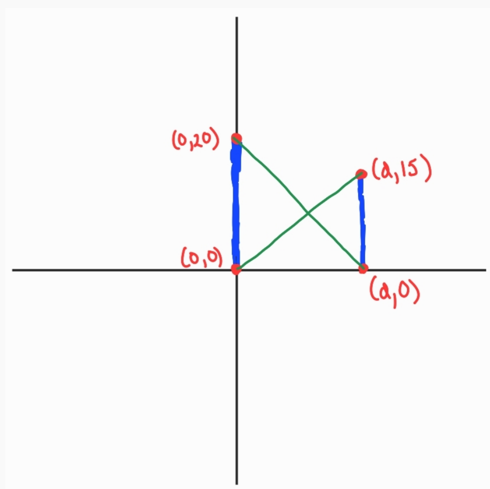
$$\begin{aligned} M_{AB} &= \left(\frac{a+0}{2}, \frac{0+b}{2}\right) = (\frac{1}{2}a, \frac{1}{2}b) \\ M_{BC} &= \left(\frac{0+0}{2}, \frac{b+0}{2}\right) = (0, \frac{1}{2}b) \\ M_{AC} &= \left(\frac{a+0}{2}, \frac{0+0}{2}\right) = (\frac{1}{2}a, 0) \end{aligned}$$

We now plug these into the area formula

$$\begin{aligned} A_{\triangle M} &= \frac{1}{2}|(\frac{1}{2}a\frac{1}{2}b - \frac{1}{2}a\frac{1}{2}b - \frac{1}{2}a\frac{1}{2}b + (0)\frac{1}{2}b - (0)(0) + \frac{1}{2}a(0)| \\ &= \frac{1}{8}ab = \frac{1}{4}A_{\triangle ABC} \end{aligned}$$

6. Two radio towers are spread some distance, d, apart. One tower is 20 metres tall and the other is 15 metres tall. A rope is strung from the top of each tower to the bottom of the other tower. The ropes cross somewhere between the two towers. What height above the ground do the ropes meet?

Solution: Setting up the poles with one sitting on the the y-axis and the other d to the right, as depicted below, we see we seek the y-value of the point of intersection of two lines.



We have one line which connects $(0, 0)$ to $(d, 15)$, and another line which connects $(0, 20)$ to $(d, 0)$.

As we have the y-intercepts for both lines, it is easiest to make use of slope-intercept forms for both.

The first line thus has equation $y = \dfrac{15-0}{d-0}x + 0 = \dfrac{15}{d}x$.

The second line thus has equation $y = \dfrac{0-20}{d-0}x + 20 = -\dfrac{20}{d}x + 20$.

We now solve for the intersection point of these lines

$$\begin{aligned}\frac{15}{d}x &= -\frac{20}{d}x + 20\\ \frac{35}{d}x &= 20\\ x &= \frac{20d}{35}\end{aligned}$$

Now substituting this back into the first equation (since it is easier to work with), we obtain

$$\begin{aligned}y &= \frac{15}{d}\frac{20d}{35}\\ y &= \frac{60}{7}\end{aligned}$$

So the ropes meet $\frac{60}{7}$ metres above the ground.

7. A point A is chosen on the line $y = x + 5$ and a point B is chosen on $y = -2x + 1$. If the midpoint M of the line segment AB is $(3, 0)$, determine the coordinates of A and B.

Solution: The point A has coordinates $(a, a+5)$, while point B has coordinates $(b, -2b+1)$, where the x values are arbitrary, and the y values come from the given equations. Plugging this and the info provided in the question into the midpoint formula gives

$$\begin{aligned}(3, 0) &= \left(\frac{a+b}{2}, \frac{a+5+(-2b+1)}{2}\right)\\ (3, 0) &= \left(\frac{a+b}{2}, \frac{a-2b+6}{2}\right)\end{aligned}$$

This gives us the system of two equations in two unknowns

$$\begin{aligned}3 &= \frac{1}{2}(a+b)\\ 0 &= \frac{1}{2}(a-2b+6)\end{aligned}$$

Solving this system yields that $a = 2$ and $b = 4$. Thus we have $A(2, 7)$ and $B(4, -7)$.

8. A vertical line divides the triangle with vertices $A(0,0)$, $B(4,0)$ and $C(8,4)$ into two regions of equal area. Find the equation of the line using analytic geometry.

Solution: Using the area formula, we have that

$$\begin{aligned} A_{\triangle ABC} &= \frac{1}{2}|(8)(0) - (8)(0) - (0)(0) + (4)(0) - (4)(4) + (0)(4)| \\ &= 8 \end{aligned}$$

Now, the line connecting A to C, using slope-intercept form, is $y = \dfrac{4-0}{8-0}x + 0 = \frac{1}{2}x$.
The line conecting B to C, using point-slope form, is $y = \dfrac{4-0}{8-4}(x-4) + 0 = x - 4$.
The line connecting A to B is just the x-axis, with equation $y = 0$.
We let the vertical line be $x = k$.
There are two options for the placement of $x = k$ given that the triangle is obtuse. Either the line intersects AB and AC, or it intersects BC and AC.
We consider the first case, which means the vertical line intersects AB at $F(k,0)$ and AC at $G(k, \frac{1}{2}k)$. In this situation we need $\triangle AFG$ to have area 4. Using the area formula again, we get

$$\begin{aligned} 4 &= \frac{1}{2}|k(0) - k(0) - (0)(0) + k(0) - k\frac{1}{2}k + (0)\frac{1}{2}k| \\ 4 &= \frac{1}{4}k^2 \\ 16 &= k^2 \\ \pm 4 &= k \end{aligned}$$

Giving us $k = 4$ and $k = -4$, but we throw out $k = -4$ as this does not reside on AB.
We now consider the second case, which means the vertical line intersects BC at $M(k, k-4)$ and AC at $N(k, \frac{1}{2}k)$. In this situation, we need $\triangle CMN$ to have area 4. Using the area

formula again, we get

$$\begin{aligned}
4 &= \frac{1}{2}|k(k-4) - k(4) - (8)(k-4) + k(4) - k\frac{1}{2}k + (8)\frac{1}{2}k| \\
8 &= |k^2 - 4k - 4k - 8k + 32 + 4k - \frac{1}{2}k^2 + 4k| \\
8 &= |\frac{1}{2}k^2 - 8k + 32| \\
8 &= \frac{1}{2}|k^2 - 16k + 64| \\
16 &= |(k-8)^2| \\
16 &= (k-8)^2 \text{ since a square is always positive} \\
\pm 4 &= k - 8 \\
8 \pm 4 &= k
\end{aligned}$$

Giving us $k = 4$ and $k = 12$, but we throw out $k = 12$ as this does not reside on BC. Thus, overall, we have found just one value that works ($k = 4$), so the line has equation $x = 4$.