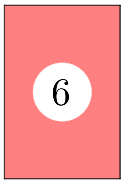
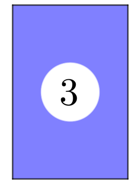


Problem of the Week
Problem C and Solution
Fair Game?



Problem

For a school mathematics project, Zesiro and Magomu created a game that uses two special decks of six cards each. The cards in one deck are labelled with the even numbers 2, 4, 6, 8, 10, and 12, and the cards in the other deck are labelled with the odd numbers 1, 3, 5, 7, 9, and 11.

A turn consists of Zesiro randomly choosing a card from the deck with even-numbered labels and Magumo randomly choosing a card from the deck with odd-numbered labels. These two cards make a pair of cards. After a pair of cards is chosen, they perform the following steps.

1. They determine the sum, S, of the numbers on the cards. For example, if Zesiro chooses the card labelled with a 6 and Magumo chooses the card labelled with a 3, then $S = 6 + 3 = 9$.

2. Using S, they determine, D, the digit sum. If S is a single digit number, then D is equal to S. If S is a two-digit number, then D is the sum of the two digits of S. For example, if Zesiro chooses the card labelled with a 6 and Magumo chooses the card labelled with a 3, then the sum and the digit sum are both 9. If Zesiro chooses the card labelled with a 10 and Magumo chooses the card labelled with a 5, then the sum is $S = 10 + 5 = 15$ and the digit sum is $D = 1 + 5 = 6$. If Zesiro chooses the card labelled with a 10 and Magumo chooses the card labelled with a 9, then the sum is $S = 10 + 9 = 19$ and the digit sum is $D = 1 + 9 = 10$.

Zesiro gets a point if the digit sum, D, is a multiple of 4.

Magomu gets a point if the number on one of the cards is a multiple of the number on the other card.

Is this game fair? That is, do Zesiro and Magomu have the same probability of getting a point on any turn? Justify your answer.

Solution

To solve this problem, we will create a table where the columns show the possible choices for the even-numbered card, the rows show the possible choices for the odd-numbered card, and each cell in the body of the table gives the sum of the corresponding pair of cards.

		Even Card					
		2	4	6	8	10	12
Odd Card	1	3	5	7	9	11	13
	3	5	7	9	11	13	15
	5	7	9	11	13	15	17
	7	9	11	13	15	17	19
	9	11	13	15	17	19	21
	11	13	15	17	19	21	23

From the table, we see that the total number of possible pairs is $6 \times 6 = 36$.

We create another table where the columns show the possible choices for the even-numbered card, the rows show the possible choices for the odd-numbered card, and each cell in the body of the table gives the digit sum of the corresponding pair of cards.

		Even Card					
		2	4	6	8	10	12
Odd Card	1	3	5	7	9	$1+1=2$	$1+3=4$
	3	5	7	9	$1+1=2$	$1+3=4$	$1+5=6$
	5	7	9	$1+1=2$	$1+3=4$	$1+5=6$	$1+7=8$
	7	9	$1+1=2$	$1+3=4$	$1+5=6$	$1+7=8$	$1+9=10$
	9	$1+1=2$	$1+3=4$	$1+5=6$	$1+7=8$	$1+9=10$	$2+1=3$
	11	$1+3=4$	$1+5=6$	$1+7=8$	$1+9=10$	$2+1=3$	$2+3=5$

If the digit sum is a multiple of 4, then Zesiro gets a point. In the table there are two digit sums, 4 and 8, that are multiples of 4. The digit sum 4 occurs six times in the table and the digit sum 8 occurs four times in the table. This totals ten possible outcomes for Zesiro, and so his probability of scoring a point on any pair is $\frac{10}{36}$.

Magomu has far less work to determine when he gets a point. None of the odd numbers are multiples of the even numbers. All multiples of even numbers are even and hence will never be odd.

Whenever a 1 is chosen, Magomu will score a point. That is, each of the six even numbers is a multiple of 1.

When a 3 is chosen, Magomu will score a point if the number on the face of the even-numbered card is a 6 or 12. That is, only two of the even numbers are multiples of 3.

When a 5 is chosen, Magomu will score a point if the number on the face of the even-numbered card is a 10. That is, only one of the even numbers is a multiple of 5.

None of the numbers in the deck containing only even numbers is a multiple of 7, 9, or 11.

So Magomu will score a point on $6 + 2 + 1 = 9$ of the 36 possible pairs. Therefore, Magomu's probability of scoring a point on any pair is $\frac{9}{36}$.

The game is not fair since Zesiro's probability of scoring a point on any pair is greater than Magomu's probability of scoring a point on any pair.