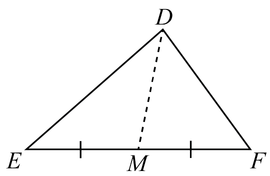


Problem of the Week

Problem C and Solution

Three Perimeters

Problem

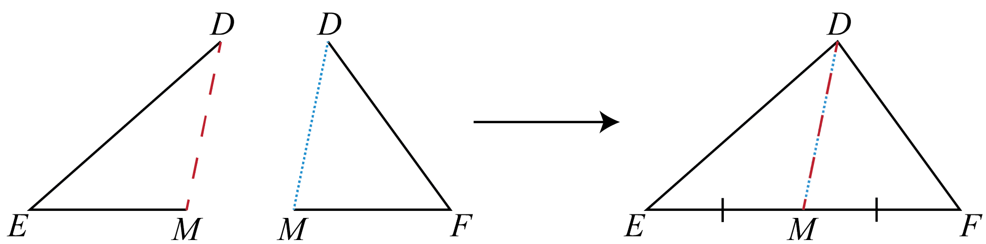
A *median* is a line segment drawn from the vertex of a triangle to the midpoint of the opposite side.

In $\triangle DEF$, a median is drawn from vertex D, meeting side EF at point M. The perimeter of $\triangle DEF$ is 24. The perimeter of $\triangle DEM$ is 18. The perimeter $\triangle DFM$ is 16. Determine the length of the median DM.

Solution

Solution 1

The perimeter of a triangle is equal to the sum of its three side lengths. Notice that the length of side EF is equal to the sum of the lengths of sides EM and MF. It follows that when we combine the perimeters of $\triangle DEM$ and $\triangle DFM$, we obtain the perimeter of $\triangle DEF$ plus two lengths of the median DM.



In other words, since the perimeter of $\triangle DEM$ is 18, the perimeter of $\triangle DFM$ is 16, and the perimeter of $\triangle DEF$ is 24, it follows that $18 + 16 = 24 + 2 \times DM$. Then $34 = 24 + 2 \times DM$, and so $2 \times DM = 10$. Therefore, the length of the median DM is 5.

Solution 2

In this solution, we take a more algebraic approach to solving the problem, using more formal equation solving.

Let $DE = t$, $EM = p$, $MF = q$, $DF = r$, and $DM = m$.

Since the perimeter of $\triangle DEM$ is 18, we can write the following equation.

$$\begin{aligned} t + p + m &= 18 \\ t + p &= 18 - m \qquad (1) \end{aligned}$$

Since the perimeter of $\triangle DFM$ is 16, we can write the following equation.

$$\begin{aligned} q + r + m &= 16 \\ q + r &= 16 - m \qquad (2) \end{aligned}$$

Since the perimeter of $\triangle DEF$ is 24, we can write the following equation.

$$t + p + q + r = 24 \qquad (3)$$

Adding equations (1) and (2) gives the following.

$$\begin{aligned} t + p &= 18 - m \qquad (1) \\ q + r &= 16 - m \qquad (2) \\ t + p + q + r &= 18 - m + 16 - m \end{aligned}$$

However from equation (3), we know that $t + p + q + r = 24$. So we can write and solve the following equation.

$$\begin{aligned} 18 - m + 16 - m &= 24 \\ 34 - 2m &= 24 \\ -2m &= 24 - 34 \\ -2m &= -10 \\ \frac{-2m}{-2} &= \frac{-10}{-2} \\ m &= 5 \end{aligned}$$

Therefore, the length of the median DM is 5.

Extension:

In the solution we never used the fact that DM is a median and that $EM = MF$. This means that there could be other triangles that satisfy the conditions of the problem without DM being the median. Indeed there are! Try creating a few different triangles with $DM = 5$ that satisfy all the conditions of the problem except the condition that DM is a median. You can do this using manipulatives, geometry software, or by hand. However, you may need some high school mathematics to calculate the precise dimensions.

It turns out that there is only one triangle that satisfies all the conditions of the problem including the fact that DM is a median.