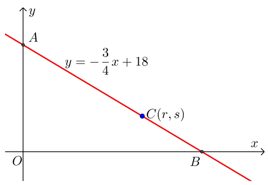


Problem of the Week

Problem D and Solution

Can You C It?

Problem

The line with equation $y = -\frac{3}{4}x + 18$ crosses the positive x-axis at point B and the positive y-axis at point A. The origin, O, and points A and B form the vertices of a triangle.

Point $C(r, s)$ lies on the line segment AB such that the area of $\triangle AOB$ is three times the area of $\triangle COB$.

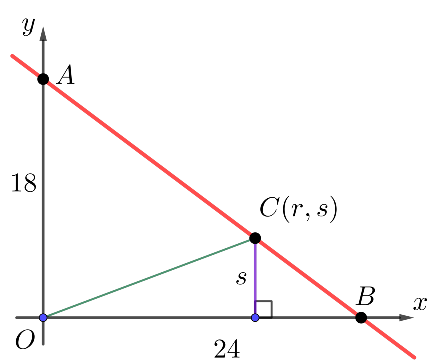
Determine the values of r and s.

Solution

The equation of the line is written in the form $y = mx + b$, where b is the y-intercept of the line. Thus, the y-intercept of the line with equation $y = -\frac{3}{4} + 18$ is 18, and $OA = 18$.

To determine the x-intercept of the line, we set $y = 0$ to obtain $0 = -\frac{3}{4}x + 18$. Solving, we have $\frac{3}{4}x = 18$, and so $x = 24$. Thus, $OB = 24$.

We drop a perpendicular from C to OB. The base of $\triangle COB$ is $OB = 24$, and since C has y-coordinate s, the height of $\triangle COB$ is s.



We now present two solutions to the problem.

Solution 1:

Since $\triangle AOB$ is a right-angled triangle with base $OB = 24$ and height $OA = 18$, using the formula area $= \frac{\text{base} \times \text{height}}{2}$, we have area of $\triangle AOB = \frac{24 \times 18}{2} = 216$.

Since the area of $\triangle AOB$ is three times the area of $\triangle COB$,
area of $\triangle COB = \frac{1}{3}(\text{area of } \triangle AOB) = \frac{1}{3}(216) = 72$.
Thus, $\triangle COB$ has area 72, base $OB = 24$, and height s.

Using the formula area $= \frac{\text{base}\times\text{height}}{2}$, we have

$$\begin{aligned}\text{area of }\triangle COB &= \frac{OB \times s}{2}\\ 72 &= \frac{24 \times s}{2}\\ 72 &= 12s\\ s &= 6\end{aligned}$$

Since $C(r,s)$ lies on the line with equation $y = -\frac{3}{4}x + 18$ and $s = 6$, we have

$$\begin{aligned}6 &= -\frac{3}{4}r + 18\\ \frac{3}{4}r &= 12\\ r &= 16\end{aligned}$$

Therefore, $r = 16$ and $s = 6$.

Solution 2:

$\triangle AOB$ and $\triangle COB$ have the same base, OB. If two triangles have the same base, then the areas of the triangles are proportional to the heights of the triangles.

Since the area of $\triangle AOB$ is three times the area of $\triangle COB$, then the height of $\triangle AOB$ is three times the height of $\triangle COB$. In other words, the height of $\triangle COB$ is $\frac{1}{3}$ the height of $\triangle AOB$.

We know that $\triangle AOB$ has height $OA = 18$ and $\triangle COB$ has height s. Therefore, $s = \frac{1}{3}(OA) = \frac{1}{3}(18) = 6$.
Since $C(r,s)$ lies on the line with equation $y = -\frac{3}{4}x + 18$ and $s = 6$, we have

$$\begin{aligned}6 &= -\frac{3}{4}r + 18\\ \frac{3}{4}r &= 12\\ r &= 16\end{aligned}$$

Therefore, $r = 16$ and $s = 6$.

Notice that in the second solution, it was actually unnecessary to find the length of OB, as this was never used.

Extension:

Can you find the coordinates of point D on line segment AB so that the area of $\triangle AOD$ is equal to the area of $\triangle COB$, thus creating three triangles of equal area? How are the points A, D, C, and B related?